

Gyromagnetic g_s factors of the spin-1/2 particles in the $(1/2^+-1/2^--3/2^-)$ triad of the four-vector spinor, ψ_μ , irreducibility, and linearity

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The gauged Klein-Gordon equation, extended by a $g_s\sigma_{\mu\nu}F^{\mu\nu}/4$ interaction, the contraction of the electromagnetic field strength tensor, $F^{\mu\nu}$, with the generators, $\sigma_{\mu\nu}/2$, of the Lorentz group in $(1/2, 0) \oplus (0, 1/2)$, and g_s being the gyromagnetic factor, is examined with the aim to find out as to what extent it qualifies as a wave equation for general relativistic spin-1/2 particles transforming as $(1/2, 0) \oplus (0, 1/2)$ and possibly distinct from the Dirac fermions. This equation can be viewed as the generalization of the $g_s = 2$ case, known under the name of the Feynman-Gell-Mann equation, the only one which allows for a bi-linearization into the gauged Dirac equation and its conjugate. At the same time, it is well known a fact that a $g_s = 2$ value can also be obtained upon the bi-linearization of the non-relativistic Schrödinger into non-relativistic Pauli equations. The inevitable conclusion is that it must not be necessarily relativity which fixes the gyromagnetic factor of the electron to $g_{(1/2)} = 2$, but rather the specific form of the primordial quadratic wave equation obeyed by it, that is amenable to a linearization. The fact is that space-time symmetries alone define solely the kinematic properties of the particles and neither fix the values of their interacting constants, nor do they necessarily prescribe linear Lagrangians. Information on such properties has to be obtained from additional physical inputs involving the dynamics. We here provide an example in support of the latter statement. Our case is that the spin-1/2⁻ fermion residing within the four-vector spinor triad, $\psi_\mu \sim (1/2^+-1/2^--3/2^-)$, whose sectors at the free particle level are interconnected by spin-up and down ladder operators, does not allow for a description within a linear framework at the interacting level. Upon gauging, despite transforming according to the irreducible $(1/2, 1) \oplus (1, 1/2)$ building block of ψ_μ , and being described by 16 dimensional four-vector spinors, though of only four independent components each, its Compton scattering cross sections, both differential and total, result equivalent to those for a spin-1/2 particle described by the generalized Feynman-Gell-Mann equation from above (for which we provide an independent algebraic motivation) and with $g_{(1/2^-)} = -2/3$. In effect, the spin-1/2⁻ particle residing within the four-vector spinor effectively behaves as a true relativistic “quadratic” fermion. The $g_{(1/2^-)} = -2/3$ value ensures in addition the desired unitarity in the ultraviolet. In contrast, the spin-1/2⁺ particle, in transforming irreducibly in the $(1/2, 0) \oplus (0, 1/2)$ sector of ψ_μ , is shown to behave as a truly linear Dirac fermion. Within the framework employed, the three spin sectors of ψ_μ are described on equal footing by representation- and spin specific wave equations and associated Lagrangians which are of second order in the momenta.

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I. INTRODUCTION

The gyromagnetic factor, g_s , is one of the fundamental constants characterizing elementary particles as it defines their magnetic moments, μ_s , and thereby the potential energy, \mathcal{V}^s , of a particle of spin- s within an external magnetic field, \mathbf{H} , given in the non-relativistic case by,

$$\mathcal{V}^s = -\mu_s \cdot \mathbf{H}, \quad \mu_s = g_s \mu_B \mathbf{S}, \quad \mu_B = \frac{e\hbar}{2mc}. \quad (1.1)$$

Here, \mathbf{S} stands for the particle's spin, m is its mass, e denotes the electric charge, μ_B is the associated elementary magneton, and g_s is the gyromagnetic factor. From now onward the physical quantities will be given in units of $\hbar = 1$, $c = 1$. Especially for the electron, this factor takes the value of $g_{(1/2+)} = 2$. This particular g_s value is closely related to the linearity of the wave equation describing the electron's propagation [1]. For a relativistic electron described by means of the Dirac equation coupled to the electromagnetic field,

$$(iD^\mu \gamma_\mu - m) \psi^D(x) = 0, \quad D^\mu = \partial^\mu + ieA^\mu, \quad [D^\mu, D^\nu] = ieF_{\mu\nu}, \quad (1.2)$$

where D^μ is the covariant derivative, A^μ is the electromagnetic gauge field, $F_{\mu\nu}$ is the electromagnetic field strength tensor, while $\psi^D(x)$ stands for the gauged Dirac spinor, the $g_{(1/2+)} = 2$ value finds a natural explanation upon squaring (1.2). In so doing, one finds the Klein-Gordon equation coupled to

the electromagnetic field in the following particular way,

$$(-i\gamma^\mu D_\mu - m)(i\gamma^\nu D_\nu - m)\psi^D(x) = -\left[(\partial_\mu + ieA_\mu)^2 + 2\left(\frac{e}{4}\right)\sigma^{\mu\nu}F_{\mu\nu} + m^2\right]\psi^D(x) = 0, \quad (1.3)$$

$$2\left(\frac{e}{4}\right)\sigma^{\mu\nu}F_{\mu\nu} = -2\begin{pmatrix} e\frac{\sigma}{2} \cdot (\mathbf{H} + i\mathbf{E}) & 0 \\ 0 & e\frac{\sigma}{2} \cdot (\mathbf{H} - i\mathbf{E}) \end{pmatrix}, \quad (1.4)$$

implying $g_{(1/2+)} = 2$. The latter equation shows that the interacting Klein-Gordon equation can be bi-linearized into the gauged Dirac equation and its conjugate, only if it has the particular form given in (1.3), i.e. exclusively for $g_{(1/2+)} = 2$. The equation (1.3) is known under the name of the Feynman-Gell-Mann equation [2]. In a similar way, the linearization of the electromagnetically coupled Schrödinger equation to the Pauli equation leads to that very same $g_{(1/2+)}$ value [1].

We here draw attention to the fact that Eq. (1.3) represents a special case of the more general equation,

$$\left[(\partial_\mu + ieA_\mu)^2 + g_s\left(\frac{e}{4}\right)\sigma^{\mu\nu}F_{\mu\nu} + m^2\right]\psi(x) = 0, \quad (1.5)$$

where g_s can be any arbitrary real constant, and if $g_s \neq 2$, then $\psi(x) \neq \psi^D(x)$. Such an equation, to be referred to as “generalized Feynman–Gell–Mann equation” has been obtained in [3] along the line of the technique of the Poincaré covariant projector method developed in [4] from considering the eigenvalue problem of the squared Pauli-Lubanski vector operator, $\omega^2(p)$, in the fundamental representation, $(1/2, 0) \oplus (0, 1/2)$ of the Lorentz algebra $sl(2, C) \sim so(1, 3)$. The $\omega_\alpha(p)$ vector expresses as,

$$\omega_\alpha(p) = \frac{1}{2}\epsilon_{\alpha\beta\rho\eta}M^{\beta\rho}p^\eta, \quad (1.6)$$

where $M^{\beta\rho}$ stand for the $so(1, 3)$ generators in $(1/2, 0) \oplus (0, 1/2)$, given by,

$$M^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\sigma^{ij}, \quad M^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{1}{2}\sigma^{0i}, \quad i, j = 1, 2, 3, \quad (1.7)$$

where γ^μ are the Dirac matrices. The squared Pauli-Lubanski vector is then calculated as,

$$\omega^2(p) = -\frac{1}{4}\sigma_{\lambda\mu}\sigma^\lambda{}_\nu p^\mu p^\nu = -\frac{3}{4}p^2 + \frac{1}{4}m\gamma_\eta\gamma_\lambda[p^\lambda, p^\eta]. \quad (1.8)$$

Its eigenvalues on, say, the momentum-space degrees of freedom, $\phi_i(p) \in (1/2, 0) \oplus (0, 1/2)$, are

$$\omega^2(p)\phi_i(p) = -m^2\frac{1}{2}\left(\frac{1}{2} + 1\right)\phi_i(p), \quad i = 1, 2, 3, 4. \quad (1.9)$$

The latter equation is equivalently cast into the form of a covariant spin-1/2- and mass m projector, $\mathcal{P}_{\omega^2}^{(m, 1/2)}(p)$ (termed to as Poincaré projector) according to,

$$\begin{aligned} \mathcal{P}_{\omega^2}^{(m, 1/2)}(p)\phi_i(p) &= \phi_i(p), \\ \mathcal{P}_{\omega^2}^{(m, 1/2)}(p) &= \frac{p^2}{m^2}\left(\frac{\omega^2(p)}{-\frac{1}{2}\left(\frac{1}{2} + 1\right)p^2}\right). \end{aligned} \quad (1.10)$$

In contrast to the standard way, in which one diagonalizes the covariant parity projector, ending up with the Dirac equation, the $\omega^2(p)$ eigenstates are of unspecified parity. However, because $\omega^2(p)$ commutes with the parity operator, the quantum number of parity always can be incorporated into the i label of $\phi_i(p)$ at a later stage. Same is valid for the polarization index, λ .

The corresponding differential equation is then obtained under the replacement, $p_\mu \rightarrow i\partial_\mu$, amounting to,

$$\left(\partial^2 - g_s\frac{i}{4}\sigma_{\mu\nu}[\partial^\mu, \partial^\nu] + m^2\right)\psi(x) = 0. \quad (1.11)$$

In the latter, the $so(1,3)$ algebraically prescribed anti-symmetric $[\partial^\mu, \partial^\nu]$ -piece of the wave equation is weighted by an arbitrary factor, g_s , with the motivation that it does not contribute to the free kinematic at all and its weight can not be uniquely fixed. The associated Lagrangian then reads,

$$\mathcal{L}_{KG}^{(1/2)} = (\partial^\mu \bar{\psi}(x)) \left(g_{\mu\nu} - g_s \frac{i}{2} \sigma_{\mu\nu} \right) \partial^\nu \psi(x) + m^2 \bar{\psi}(x) \psi(x). \quad (1.12)$$

The equations (1.11) and (1.12) provide the most general framework for relativistic spin-1/2 description in so far as they allow for the existence of truly “quadratic” fermions characterized by a gyromagnetic factor $g_{(1/2)} \neq 2$.

This consideration shows that, as already argued in [1], the $g_{(1/2)} = 2$ value for the electron is not due to relativity alone but also to the form of the quadratic equation obeyed by it, which permits a bi-linearization, be the equation relativistic or not. The fact that relativity alone does not fix the values of the gyromagnetic ratio has also been noticed in [4], where the gyromagnetic factor of spin-3/2⁻ within the four-vector-spinor has been fixed to $g_{(3/2-)} = 2$ from the requirement on causal propagation within an electromagnetic environment, and for the case of a Lagrangian of second order in the momenta (see Appendix II for a more detailed coverage of this issue).

The goal of the present study is to provide a consistent and gauge invariant description of the spin degrees of freedom in the four-vector spinor space, to show that the spin-1/2⁻ companion to spin-3/2⁻ within the $(1/2^+ - 1/2^- - 3/2^-)$ triad behaves in Compton scattering as a “quadratic” spin-1/2 fermion, and to explore consequences.

The article is structured as follows. In the next section we present first the conventional complete set of the degrees of freedom (d.o.f.) spanning the four-vector-spinor space and the associated auxiliary conditions through which they are unambiguously identified. We conclude on the impossibility of a linear description of the spin-1/2⁻ residing there, and furthermore notice that both the spin-1/2⁺ and spin-1/2⁻ sectors are reducible with respect to Lorentz transformations. Through the text, we denote the reducibility, or the irreducibility with respect to Lorentz transformations by “ $so(1,3)$ reducibility/irreducibility”. Motivated by the interconnection of the ψ_μ degrees of freedom by spin-up and down ladder operators, we search for a possibility to describe all the spin degrees of freedom on equal footing, and test a particular second order formalism, highlighted in Section 3. Namely, we present there the recently developed general reduction algorithm of Lorentz algebra representations [5] based on the one of the momentum independent Casimir invariants of the Lorentz algebra and apply it to the four-vector spinor with the task to split the irreducible Dirac sector, $(1/2, 0) \oplus (0, 1/2)$, from the genuine irreducible Rarita-Schwinger sector, $(1/2, 1) \oplus (1, 1/2)$. Also there, we perform the separation between the non-interacting d.o.f.’s of the spin-1/2⁻ and 3/2⁻ residents of $(1/2, 1) \oplus (1, 1/2)$, employing covariant spin-projectors based upon the squared Pauli-Lubanski vector operator, a Casimir invariant of the Poincaré algebra, and present the free particle wave equations and Lagrangians obtained in this way, which are all representation- and spin specific, and of second order in the momenta. The wave equations for the spin-1/2 four-vector-spinors can be reduced from 16 to 4 degrees of freedom through contractions either by p^α , or γ^α . The equations obtained in this way are especially simple, as they appear to be of the type given in (1.11) above. In Section 4 we couple the aforementioned contracted wave equations to the electromagnetic field and show that they take each the form of the generalized Feynman–Gell-Mann equation (1.5). Section 5 is devoted to the gauged equations and associated Lagrangians in the full 16 dimensional space, from which we read off the electromagnetic current densities. Furthermore, we calculate in same space the magnetic dipole moments for spin-1/2⁺ in $(1/2, 0) \oplus (0, 1/2)$, and spin-1/2⁻ in $(1/2, 1) \oplus (1, 1/2)$, and find them fixed to $g_{(1/2+)} = 2$, and $g_{(1/2-)} = -2/3$, respectively. These are the values which also show up in the contracted equations. With the aim to check whether the contracted and full-space gauged equations are equivalent, we calculate in section 6 the differential and total Compton scattering cross sections off spin-1/2⁻ first with the vertexes from the gauged Lagrangians in the full space. Next we compare them with the results alternatively obtained with the vertexes following from the contracted equation, i.e the one that describes this sector in terms of the correct number of four gauged spinorial degrees of freedom, and is shaped after (1.5) with $g_{(1/2-)} = -2/3$. We encounter an exact coincidence, meaning that the description of Compton scattering off this target provided by the contracted gauged spin-1/2⁻ wave equation is indeed equivalent to the description provided by the equation in the full space of the gauged four-vector-spinor degrees of freedom. This observation allows us first to conclude that the

spin- $1/2^-$ resident in ψ_μ behaves effectively as a truly “quadratic” fermion, and then that the suggested covariant separation prescription of the spin- $1/2^-$ and spin- $3/2^-$ degrees of freedom in the irreducible $(1/2, 1) \oplus (1, 1/2)$ building block of the four-vector–spinor remains valid upon gauging. In contrast, the spin- $1/2^+$ is shown to be a truly linear Dirac fermion. The paper closes with brief conclusions. It has one appendix containing the spin-up and down ladder operators within the four-vector spinor space, and a second one in which we briefly review for the sake of the self sufficiency of the presentation the here omitted spin- $3/2^-$ degrees of freedom, previously extensively studied elsewhere.

II. THE CONVENTIONAL DEGREES OF FREEDOM SPANNING THE FOUR-VECTOR SPINOR SPACE AND THEIR IDENTIFICATION THROUGH AUXILIARY CONDITIONS

The four-vector spinor, ψ_μ , is the direct product of the four vector, $(1/2, 1/2)$ and the Dirac spinor, $(1/2, 0) \oplus (0, 1/2)$. This representation space of the Lorentz algebra, $so(1, 3)$, is reducible with respect to Lorentz transformations as,

$$\psi_\mu \sim \left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right] \longrightarrow \left[\left(\frac{1}{2}, 1\right) \oplus \left(1, \frac{1}{2}\right)\right] \oplus \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right]. \quad (2.1)$$

Accordingly, the number of the degrees of freedom is sixteen, out of which twelve belong to $(1/2, 1) \oplus (1, 1/2)$, while the remaining four are contained in $(1/2, 0) \oplus (0, 1/2)$. If the Dirac particle is considered to be of positive parity, then the spin and parity content of the first irreducible representation space is $3/2^-$, and $1/2^-$. Usually, one constructs the degrees of freedom in momentum space, an exercise frequently performed in the literature, among others in [6], [4]. The $(1/2, 1/2)$ representation is spanned by one time-like scalar, $[\eta_0(\mathbf{p}, 0)]^\alpha$, and three space-like (spin- 1^-) degrees of freedom, $[\eta(\mathbf{p}, \ell)]^\alpha$ (with $\ell = -1, 0, 1$). The spin- $1/2^+$ four-vector spinors emerging from the coupling of the scalar in $(1/2, 1/2)$ to the Dirac spinor will be termed to as Scalar-Spinors (**SS**), and denoted by $[\mathcal{U}_\pm^{\text{SS}}(\mathbf{p}, 1/2, \lambda)]^\alpha$, finding

$$\left[\mathcal{U}_\pm^{\text{SS}}(\mathbf{p}, \frac{1}{2}, \lambda)\right]^\alpha = [\eta_0(\mathbf{p}, 0)]^\alpha u_\pm(\mathbf{p}, \lambda) = \frac{p^\alpha}{m} u_\pm(\mathbf{p}, \lambda). \quad (2.2)$$

Here, $[\eta_0(\mathbf{p}, 0)]^\alpha = p^\alpha/m$ with $p^2 = m^2$, is the only spin- 0^+ vector in $(1/2, 1/2)$, where the parity operator is given by the metric tensor $g_{\mu\nu}$, while $u_\pm(\mathbf{p}, \lambda)$ are the usual Dirac particle-antiparticle spinors in $(1/2, 0) \oplus (0, 1/2)$, of positive (+) and negative (−) parities, respectively, (i.e., the u - and v -spinors in the terminology of [7]), and whose polarizations are $\lambda = 1/2, -1/2$. The spin- $1/2^-$ vector-spinors emerge through the coupling of the spin- 1^- four-vectors $[\eta(\mathbf{p}, \ell)]^\alpha$ [4], to the Dirac spinor and will be termed to as Vector-Spinors (**VS**). They are commonly constructed in the standard way by employing the ordinary angular momentum coupling scheme and in terms of appropriate Clebsch- Gordan coefficients,

$$\left[\mathcal{U}_\mp^{\text{VS}}\left(\mathbf{p}, \frac{1}{2}, +\frac{1}{2}\right)\right]^\alpha = -\sqrt{\frac{1}{3}}[\eta(\mathbf{p}, 0)]^\alpha u_\pm\left(\mathbf{p}, +\frac{1}{2}\right) + \sqrt{\frac{2}{3}}[\eta(\mathbf{p}, +1)]^\alpha u_\pm\left(\mathbf{p}, -\frac{1}{2}\right), \quad (2.3a)$$

$$\left[\mathcal{U}_\mp^{\text{VS}}\left(\mathbf{p}, \frac{1}{2}, -\frac{1}{2}\right)\right]^\alpha = \sqrt{\frac{1}{3}}[\eta(\mathbf{p}, 0)]^\alpha u_\pm\left(\mathbf{p}, -\frac{1}{2}\right) - \sqrt{\frac{2}{3}}[\eta(\mathbf{p}, -1)]^\alpha u_\pm\left(\mathbf{p}, +\frac{1}{2}\right). \quad (2.3b)$$

Notice that if $[\eta_0(\mathbf{p}, 0)]^\alpha$ is chosen to be of positive parity, as done here, the parity of $[\eta(\mathbf{p}, \ell)]^\alpha$ is negative. In consequence, the positive- (negative-) parity scalar-spinors are made up of positive- (negative-) parity Dirac spinors, while the positive- (negative-) parity vector-spinors are made up of negative- (positive-) parity Dirac spinors. Finally, the spin- $3/2^-$ degrees of freedom are given by the following widely spread

Clebsch-Gordan combinations, which we here list solely for the sake of self-sufficiency of the presentation:

$$\left[\mathcal{U}_{\mp} \left(\mathbf{p}, \frac{3}{2}, +\frac{3}{2} \right) \right]^{\alpha} = [\eta(\mathbf{p}, +1)]^{\alpha} u_{\pm} \left(\mathbf{p}, +\frac{1}{2} \right), \quad (2.4)$$

$$\left[\mathcal{U}_{\mp} \left(\mathbf{p}, \frac{3}{2}, +\frac{1}{2} \right) \right]^{\alpha} = \frac{1}{\sqrt{3}} [\eta(\mathbf{p}, +1)]^{\alpha} u_{\pm} \left(\mathbf{p}, -\frac{1}{2} \right) + \sqrt{\frac{2}{3}} [\eta(\mathbf{p}, 0)]^{\alpha} u_{\pm} \left(\mathbf{p}, +\frac{1}{2} \right), \quad (2.5)$$

$$\left[\mathcal{U}_{\mp} \left(\mathbf{p}, \frac{3}{2}, -\frac{1}{2} \right) \right]^{\alpha} = \frac{1}{\sqrt{3}} [\eta(\mathbf{p}, -1)]^{\alpha} u_{\pm} \left(\mathbf{p}, +\frac{1}{2} \right) + \sqrt{\frac{2}{3}} [\eta(\mathbf{p}, 0)]^{\alpha} u_{\pm} \left(\mathbf{p}, -\frac{1}{2} \right), \quad (2.6)$$

$$\left[\mathcal{U}_{\mp} \left(\mathbf{p}, \frac{3}{2}, -\frac{3}{2} \right) \right]^{\alpha} = [\eta(\mathbf{p}, -1)]^{\alpha} u_{\pm} \left(\mathbf{p}, -\frac{1}{2} \right). \quad (2.7)$$

They have been extensively studied in the literature beginning with [8], have been highlighted in several textbooks [9], [10], and will be as a rule left aside from the main body of the manuscript, though we comment on them in the Appendix B. In effect, one first finds the four independent scalar-spinors, $[\mathcal{U}_{\pm}^{\text{SS}}(\mathbf{p}, 1/2, \lambda)]^{\alpha}$, in (2.2), and then as another set, the four independent vector-spinors, $[\mathcal{U}_{\pm}^{\text{VS}}(\mathbf{p}, 1/2, \lambda)]^{\alpha}$, in (2.3a)-(2.3b) summing up to eight (including anti-particles) spin-1/2 $^{\pm}$ states, as it should be. Together with the eight independent spin-3/2 $^{-}$ states (as a rule not to be considered here further) the total of sixteen independent degrees of freedom in the four-vector-spinor is recovered. We like to emphasize that the degrees of freedom are interconnected through spin-up and down ladder operators, as summarized in the Appendix I, a circumstance that strongly suggests to consider all of them on equal footing. In the next section we will show that the degrees of freedom given in the equations (2.2), (2.3a), (2.3b), behave reducible under Lorentz transformations.

The form of the spin-1/2 $^{-}$ vector-spinors can be significantly simplified by means of the Pauli-Lubanski vectors, $\mathcal{W}^{\mu}(p)$, in the four-vector spinor, on the one side, and the Pauli-Lubanski vector, $\omega^{\mu}(p)$, in the Dirac-spinor, on the other side. The general expression for a Pauli-Lubanski vector is given by [11],

$$[\mathcal{W}_{\lambda}(p)]_{AB} = \frac{1}{2} \epsilon_{\lambda\rho\sigma\mu} [M^{\rho\sigma}]_{AB} p^{\mu}, \quad (2.8)$$

where $M^{\rho\sigma}$ are the generators of the Lorentz algebra in the representation space of interest, while A , and B are the sets of indexes that completely characterize its dimensionality. Specifically for the four-vector spinor the generators, $[M_{\mu\nu}^{\text{V}}]_{\alpha\beta}$ and $[M_{\mu\nu}^{\text{S}}]$, within the respective Four-Vector-, and the Dirac-Spinor building blocks read,

$$[M_{\mu\nu}^{\text{V}}]_{\alpha\beta} = i(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}), \quad (2.9)$$

$$[M_{\mu\nu}^{\text{S}}]_{ab} = \frac{1}{2} [\sigma_{\mu\nu}]_{ab} = \frac{i}{4} [\gamma_{\mu}, \gamma_{\nu}]_{ab}, \quad (2.10)$$

$$[M_{\mu\nu}]_{(\alpha a), (\beta b)} = [M_{\mu\nu}^{\text{V}}]_{\alpha\beta} \delta_{ab} + g_{\alpha\beta} [M_{\mu\nu}^{\text{S}}]_{ab}, \quad (2.11)$$

with γ_{μ} being the standard Dirac matrices. Notice that the operators in (2.11) are (16×16) matrices, as should be the generators in ψ_{μ} , and consequently carry next to the Lorentz indexes, also Dirac spinor indexes, $(a, b..)$, here denoted by small Latin letters. With that, $\mathcal{W}_{\mu}(p)$ expresses as the direct sum of the Pauli-Lubanski vectors, $W_{\mu}(p)$, and $\omega_{\mu}(p)$, in the respective $(1/2, 1/2)$ - and Dirac-building blocks according to [4],

$$\begin{aligned} [\mathcal{W}_{\mu}(p)]_{(\alpha a), (\beta b)} &= [\omega_{\mu}(p)]_{ab} g_{\alpha\beta} + [W_{\mu}(p)]_{\alpha\beta} \delta_{ab}, \\ [\omega_{\mu}(p)]_{ab} &= -\frac{1}{2} [\gamma_5(p_{\mu} - \gamma_{\mu}\not{p})]_{ab}, \quad [W_{\mu}(p)]_{\alpha}{}^{\beta} = i\epsilon_{\mu\alpha}{}^{\beta}{}_{\sigma} p^{\sigma}. \end{aligned} \quad (2.12)$$

In reference to (2.8), the label A in the case under consideration consists of a Lorentz index, α , and a spinor index, a , i.e. A presents itself as a double (four-vector)-(Dirac-spinor) index, $A \simeq (\alpha a)$. Finally, the squared Pauli-Lubanski vector in the four-vector- spinor is calculated as,

$$[\mathcal{W}^2(p)]_{\alpha\beta} = [\mathcal{W}^{\sigma}(p)]_{\alpha}{}^{\gamma} [\mathcal{W}_{\sigma}(p)]_{\gamma\beta} \quad (2.13)$$

$$= \frac{1}{4} \epsilon^{\sigma}{}_{\lambda\tau\mu} \epsilon_{\sigma\eta\zeta\nu} [M^{\lambda\tau}]_{\alpha}{}^{\delta} [M^{\eta\zeta}]_{\delta\beta} p^{\mu} p^{\nu}. \quad (2.14)$$

where the Dirac spinor indexes have been suppressed for the sake of simplifying the notations. The squared Pauli-Lubanski vector operator can be diagonalized within the four-vector spinor space as,

$$[\mathcal{W}^2(p)]^\alpha{}_\beta \psi^\beta = -p^2 s_i (s_i + 1) \psi^\alpha, \quad i = 1, 2, \quad s_1 = \frac{1}{2}, \quad s_2 = \frac{3}{2}. \quad (2.15)$$

After some algebra, the following subtle relationship between the squared Pauli-Lubanski vector $\mathcal{W}^2(p)$ in Eqs. (2.13)-(2.14), on the one side, and the Pauli-Lubanski vector $\omega_\alpha(p)$ in the Dirac- representation (2.12),

$$[\mathcal{W}^2(p)]_\alpha{}^\beta \omega_\beta(p) = -p^2 \frac{1}{2} \left(\frac{1}{2} + 1 \right) \omega_\alpha(p), \quad (2.16)$$

can be verified. The latter equation indicates that the spin-1/2⁻ degrees of freedom in ψ_μ can be equivalently re-expressed by the aid of the Pauli-Lubanski vector, $\omega_\alpha(p)$, from the Dirac building block. Indeed, one can easily verify component by component that the spin-1/2⁻ states in the equations (2.3a) and (2.3b) are equivalently cast as,

$$\begin{aligned} \left[\mathcal{U}_\pm^{\text{VS}} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\alpha &= \frac{2}{\sqrt{3}m} \omega^\alpha(p) \gamma^5 u_\pm(\mathbf{p}, \lambda) \\ &= \frac{2}{\sqrt{3}m} \left(\frac{1}{2} \gamma^5 (-p^\alpha + \gamma^\alpha \not{p}) \right) \gamma^5 u_\pm(\mathbf{p}, \lambda) \\ &= \frac{1}{\sqrt{3}m} (-p^\alpha + \gamma^\alpha \not{p}) u_\pm(\mathbf{p}, \lambda). \end{aligned} \quad (2.17)$$

The expressions in (2.2) and (2.17) will prove very useful in the following. Specifically, the demonstration of the orthogonality between scalar- and vector-spinors will benefit from the well known property of the Pauli-Lubanski vectors of being divergence-less,

$$p^\alpha \omega_\alpha(p) = 0, \quad (2.18)$$

which implies that the vector- spinors satisfy the following auxiliary conditions:

$$p^\alpha \left[\mathcal{U}_\pm^{\text{VS}} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_\alpha = 0, \quad (2.19)$$

$$(\not{p} \pm m) \omega^\alpha(p) \left[\mathcal{U}_\pm^{\text{VS}} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_\alpha = 0. \quad (2.20)$$

However, the first condition is not exclusive to the spin-1/2⁻ particle but is also shared by the spin-3/2⁻ degrees of freedom, characterized by the well known set of auxiliary conditions [8],[10],

$$p^\alpha \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \lambda \right) \right]_\alpha = 0, \quad (2.21)$$

$$\gamma^\alpha \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \lambda \right) \right]_\alpha = 0. \quad (2.22)$$

Instead, $[\mathcal{U}_\pm^{\text{SS}}(\mathbf{p}, 1/2, \lambda)]^\alpha$ is uniquely identified by,

$$\mathcal{W}^\alpha(p) [\mathcal{U}_\pm^{\text{SS}}(\mathbf{p}, \lambda)]_\alpha = 0, \quad (2.23)$$

an observation due to [6]. All three different spin-parity sectors obey the Dirac equation:

$$(\not{p} \pm m) \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \lambda \right) \right]_\alpha = (\not{p} \pm m) \left[\mathcal{U}_\pm^{\text{VS}} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_\alpha = (\not{p} \mp m) \left[\mathcal{U}_\pm^{\text{SS}} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_\alpha = 0. \quad (2.24)$$

In consequence, anyone of the reducible degrees of freedom of the four-vector-spinor satisfies the Dirac equation and can be uniquely characterized by its own set of auxiliary conditions [6]. As already

$so(1,3)$ reducible free spin-1/2 degrees of freedom	Wave equation and auxiliary condition(s)
spin-1/2 ⁺ : $[\mathcal{U}_{\pm}^{\text{SS}}(\mathbf{p}, \frac{1}{2}, \lambda)]^{\alpha} = \frac{p^{\alpha}}{m} u_{\pm}(\mathbf{p}, \lambda), \quad \text{eq. (2.2)}$	1. $(\not{p} \mp m) [\mathcal{U}_{\pm}^{\text{SS}}(\mathbf{p}, \frac{1}{2}, \lambda)]^{\alpha} = 0 \quad \text{eq. (2.24)}$ 2. $\mathcal{W}^{\alpha}(p) [\mathcal{U}_{\pm}^{\text{SS}}(\mathbf{p}, \frac{1}{2}, \lambda)]_{\alpha} = 0 \quad \text{eq. (2.23)}$
spin-1/2 ⁻ : $[\mathcal{U}_{\pm}^{\text{VS}}(\mathbf{p}, \frac{1}{2}, \lambda)]^{\alpha} = \frac{2}{\sqrt{3}m} \omega^{\alpha}(p) \gamma^5 u_{\mp}(\mathbf{p}, \lambda)$ eqs. (2.3)-(2.17)	1. $(\not{p} \pm m) [\mathcal{U}_{\pm}^{\text{VS}}(\mathbf{p}, \frac{1}{2}, \lambda)]^{\alpha} = 0 \quad \text{eq. (2.24)}$ 2. $(\not{p} \pm m) \omega^{\alpha}(p) [\mathcal{U}_{\pm}^{\text{VS}}(\mathbf{p}, \frac{1}{2}, \lambda)]_{\alpha} = 0$ eq. (2.20) 3. $p^{\alpha} [\mathcal{U}_{\pm}^{\text{VS}}(\mathbf{p}, \frac{1}{2}, \lambda)]_{\alpha} = 0 \quad \text{eq. (2.19)}$

TABLE I: Identification of the $so(1,3)$ reducible spin-1/2⁺ and spin-1/2⁻ degrees of freedom within the four-vector spinor (left column) through the Dirac equation and proper auxiliary conditions (right column).

announced in the previous section, from now on we focus on the two spin-1/2[±] sectors in ψ_{μ} , summarized in Table 1. It is visible from eq. (2.20) and the Table 1 that the auxiliary condition identifying the spin-1/2⁻ sector is of second order in the momenta. This shows that its degrees of freedom are lacking a description within the linear Rarita-Schwinger framework. The linear framework exploited in the description of the highest spin-3/2⁻ is anyway widely known to be plagued by several inconsistency problems [12]. In one of the possibilities such problems have been circumvented within the second order theory of the Poincaré covariant projectors developed in [4], a framework which we partly follow.

III. THE $so(1,3)$ IRREDUCIBLE DEGREES OF FREEDOM IN THE FOUR-VECTOR-SPINOR SPACE AND A UNIFIED SECOND ORDER DESCRIPTION FREE FROM AUXILIARY CONDITIONS

The goal of the current section is to design a description of all the degrees of freedom spanning the four-vector spinor space on equal footing. This could be of interest in studying in physical processes possible interferences between the highest spin-3/2⁻, with the lower-spin components (see [13] for a related discussion). Such turns out to be realizable within a second order formalism, and with the special emphasis on the strictly $so(1,3)$ irreducible d.o.f. As we shall see in due places, the approach will be free from auxiliary conditions. The section is structured as follows. In the first subsection we highlight a recently developed $so(1,3)$ representation reduction algorithm [5] based on static projectors constructed from one of the Casimir invariants of the Lorentz algebra. In the second subsection we identify the $so(1,3)$ irreducible $(1/2, 0) \oplus (0, 1/2)$ sector in the four-vector spinor and split it neatly from $(1/2, 1) \oplus (1, 1/2)$, which now exclusively hosts the spin-1/2⁻ and 3/2⁻. In the third subsection we construct along the line of ref. [4] covariant mass- m and spin-1/2 projectors from the Casimir invariants of the Poincaré algebra, i.e. from the squared four-momentum, P^2 , and the squared Pauli-Lubanski vector operator, $\mathcal{W}^2(p)$, which are of second order in the momenta. Finally, in the fourth subsection we combine the aforementioned Lorentz- and Poincaré projectors and obtain the second order wave equations satisfied by the two spin-1/2 degrees of freedom from the irreducible $so(1,3)$ sectors in the four vector spinor.

A. Recognizing $so(1,3)$ irreducible representation spaces by static projectors derived from an invariant of the Lorentz algebra

To ensure that each one of the two spin- $1/2^+$ and spin- $1/2^-$ degrees of freedom of ψ_μ transforms exclusively according to one of its two $so(1,3)$ irreducible sectors, i.e. to either $(1/2, 0) \oplus (0, 1/2)$, or, $(1/2, 1) \oplus (1, 1/2)$, and thereby to pay a tribute to Wigner's particle definition [14], it is necessary to introduce projectors based upon one of the Casimir invariants of the Lorentz algebra. The Lorentz algebra has a Casimir operator, denoted by F , and given by [11],

$$[F]_{\alpha\beta} = \frac{1}{4}[M^{\mu\nu}]_{\alpha}{}^{\gamma}[M_{\mu\nu}]_{\gamma\beta} = \frac{9}{4}g_{\alpha\beta} + \frac{i}{2}\sigma_{\alpha\beta}, \quad (3.1)$$

where use has been made of (2.10)–(2.9), and (2.11) to work out the explicit expression on the rhs in (3.1). Its eigenvalue problem reads,

$$F|j_1, j_2\rangle = \frac{1}{2}(K(K+2) + M^2)|j_1, j_2\rangle = \frac{1}{2}(j_1(j_1+1) + j_2(j_2+1))|j_1, j_2\rangle, \quad (3.2)$$

where $|j_1, j_2\rangle$ stand for some generic states transforming irreducibly under $so(1,3)$ as $(j_2, j_1) \oplus (j_1, j_2)$, while

$$K = j_1 + j_2, \quad M = |j_1 - j_2|. \quad (3.3)$$

The K - and M -values fully characterize any $(j_2, j_1) \oplus (j_1, j_2)$ representation space, remain same in all inertial frames, and allow, if preferred, for the following relabeling,

$$(j_2, j_1) \oplus (j_1, j_2) \simeq (K, M). \quad (3.4)$$

On the basis of F one can construct projector operators on the irreducible sectors, $(1/2, j_1) \oplus (j_1, 1/2)$, in ψ_μ , with $j_1 = 0, 1$. Such projectors will be denoted hereafter by, $\mathcal{P}_F^{(1/2, j_1)}$, and we find them as,

$$\mathcal{P}_F^{(1/2, 0)} = \left(\frac{F - \lambda_1}{\lambda_0 - \lambda_1} \right), \quad (3.5)$$

$$\mathcal{P}_F^{(1/2, 1)} = \left(\frac{F - \lambda_0}{\lambda_1 - \lambda_0} \right), \quad (3.6)$$

where, λ_{j_1} are the F eigenvalues,

$$\lambda_0 = \frac{3}{4}, \quad \lambda_1 = \frac{11}{4}, \quad (3.7)$$

corresponding to $j_2 = 1/2$ and $j_1 = 0, 1$, respectively. Their respective momentum-space eigenstates, denoted by $w^{(1/2, j_1)}(\mathbf{p}, s_i, \lambda)$, transform accordingly as the $(1/2, j_1) \oplus (j_1, 1/2)$ sectors of the four-vector–spinor representation space:

$$j_1 = 0, \quad \left(\frac{1}{2}, 0 \right) \oplus \left(0, \frac{1}{2} \right) : \quad \mathcal{P}_F^{(1/2, 0)} w^{(1/2, 0)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) = w^{(1/2, 0)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right), \quad (3.8)$$

$$j_1 = 1, \quad \left(\frac{1}{2}, 1 \right) \oplus \left(1, \frac{1}{2} \right) : \quad \mathcal{P}_F^{(1/2, 1)} w^{(1/2, 1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) = w^{(1/2, 1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right). \quad (3.9)$$

The respective Lorentz projectors are then easily calculated as,

$$[\mathcal{P}_F^{(1/2, 0)}]_{\alpha\beta} = \frac{1}{4}\gamma_\alpha\gamma_\beta, \quad (3.10)$$

$$[\mathcal{P}_F^{(1/2, 1)}]_{\alpha\beta} = g_{\alpha\beta} - \frac{1}{4}\gamma_\alpha\gamma_\beta. \quad (3.11)$$

The orthogonality and completeness properties of the above set of operators are easily verified. The Lorentz projectors have the advantage to be momentum-independent, which allows one any time when they are put at work, to flawlessly recognize within a $so(1,3)$ reducible representation space its $so(1,3)$ irreducible sectors for which they have been specifically designed, and to remove the rest without increasing the power of the momentum dependence of the wave equation. The next section is devoted to the solution of the above eqs. (3.8) and (3.9).

B. Identifying the irreducible degrees of freedom in ψ_μ and equations of motion

It is straightforward to verify that none of the $[\mathcal{U}_\pm^{\text{SS}}(\mathbf{p}, 1/2, \lambda)]^\alpha$ and $[\mathcal{U}_\pm^{\text{VS}}(\mathbf{p}, 1/2, \lambda)]^\alpha$ four-vector spinors commonly used as bases within the four-vector spinor space and given in the respective Eqs. (2.2), (2.3a), and (2.3b) from above behaves irreducibly under Lorentz transformations, this because none of these states acts as an eigenstate to the F -Casimir invariant of the Lorentz algebra in (3.2). Towards our goal of finding the explicit expressions for the $so(1, 3)$ irreducible spin-1/2⁺ and 1/2⁻ states, from now onwards denoted by $[w_\pm^{(1/2, j_1)}(\mathbf{p}, 1/2, \lambda)]^\alpha$, with the low case index, \pm , specifying their respective parities (to become indicative of particle-antiparticle upon quantization) we benefit from our knowledge on $[\mathcal{U}_\pm^{\text{SS}}(\mathbf{p}, 1/2, \lambda)]^\alpha$ in (2.2) and $[\mathcal{U}_\pm^{\text{VS}}(\mathbf{p}, 1/2, \lambda)]^\alpha$ in (2.17) and find their respective $j_1 = 0$ -projections, i.e. the projections on $(1/2, 0) \oplus (0, 1/2)$, in employing the operator from (3.10) according to,

$$[\mathcal{P}_F^{(1/2, 0)}]_{\alpha\beta} \left[\mathcal{U}_\pm^{\text{SS}} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_\beta = \frac{1}{4m} \gamma^\alpha \not{p} u_\pm(\mathbf{p}, \lambda), \quad (3.12)$$

$$[\mathcal{P}_F^{(1/2, 0)}]_{\alpha\beta} \left[\mathcal{U}_\pm^{\text{VS}} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_\beta = \frac{\sqrt{3}}{4m} \gamma^\alpha \not{p} u_\pm(\mathbf{p}, \lambda). \quad (3.13)$$

In this manner the $(1/2, 0) \oplus (0, 1/2)$ components of the spin-1/2⁺ Clebsch-Gordan combinations, $[\mathcal{U}_\pm^{\text{SS}}(\mathbf{p}, 1/2, \lambda)]^\alpha$, and $[\mathcal{U}_\pm^{\text{VS}}(\mathbf{p}, 1/2, \lambda)]^\alpha$, are unambiguously identified modulo a normalization constant. In a way similar, their $(1/2, 1) \oplus (1, 1/2)$ components are identified by employing the operator in (3.11) as,

$$[\mathcal{P}_F^{(1/2, 1)}]_{\alpha\beta} \left[\mathcal{U}_\pm^{\text{SS}} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_\beta = \frac{1}{m} \left(p^\alpha - \frac{1}{4} \gamma^\alpha \not{p} \right) u_\pm(\mathbf{p}, \lambda), \quad (3.14)$$

$$[\mathcal{P}_F^{(1/2, 1)}]_{\alpha\beta} \left[\mathcal{U}_\pm^{\text{VS}} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_\beta = -\frac{1}{\sqrt{3}m} \left(p^\alpha - \frac{1}{4} \gamma^\alpha \not{p} \right) u_\pm(\mathbf{p}, \lambda). \quad (3.15)$$

Taking care of the normalizations, the non-interacting irreducible spin-1/2⁺ states, $[w_\pm^{(1/2, j_1)}(\mathbf{p}, 1/2, \lambda)]^\alpha$, can finally be cast in terms of covariant entities of factorized Dirac-and four-vector degrees of freedom, the former being given by the ordinary Dirac spinors, and the latter by p^α , and/or γ^α ,

$$\begin{aligned} \text{spin} - \frac{1}{2}^+ \in \left(\frac{1}{2}, 0 \right) \oplus \left(0, \frac{1}{2} \right) : \quad \left[w_\pm^{(1/2, 0)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\alpha &= \frac{1}{2m} \gamma^\alpha \not{p} u_\pm(\mathbf{p}, \lambda) \\ &= \pm \frac{1}{2} \gamma^\alpha u_\pm(\mathbf{p}, \lambda), \end{aligned} \quad (3.16)$$

$$\text{spin} - \frac{1}{2}^- \in \left(\frac{1}{2}, 1 \right) \oplus \left(1, \frac{1}{2} \right) : \quad \left[w_\pm^{(1/2, 1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\alpha = \frac{2}{\sqrt{3}m} \left(p^\alpha - \frac{1}{4} \gamma^\alpha \not{p} \right) u_\pm(\mathbf{p}, \lambda), \quad (3.17)$$

where use has been made of the Dirac equation. Therefore, one finds two polarizations available for each parity and two possible parities for each j_1 value, making a total of eight spin-1/2 independent states which now reside each in two distinct irreducible Lorentz invariant representation subspaces of the four-vector-spinor. The above eigenstates to the Lorentz projector are also of well defined parities, and are normalized as,

$$\left[\overline{w}_\pm^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\alpha \left[w_\pm^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_\alpha = \pm 1, \quad (3.18)$$

their conjugates being defined as,

$$\overline{w}_\pm^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) = \left[\gamma^0 w_\pm^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\dagger. \quad (3.19)$$

Finally, the highest spin-3/2⁻ degrees of freedom in the Eqs. (2.5)–(2.7) are automatically $so(1, 3)$ irreducible because they transform exclusively according $(1/2, 1) \oplus (1, 1/2)$ and imply,

$$\left[w_\pm^{(1/2, 1)} \left(\mathbf{p}, \frac{3}{2}, \sigma \right) \right]^\alpha = \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \sigma \right) \right]^\alpha. \quad (3.20)$$

C. The covariant mass- m and spin-1/2 projector from the invariants of the Poincaré algebra

As repeatedly emphasized, the $so(1, 3)$ irreducible Rarita-Schwinger sector $(1/2, 1) \oplus (1, 1/2)$ contains two spins, namely $s_1 = 1/2^-$ and $s_2 = 3/2^-$, and one can construct covariant mass- m and spin- s_i projectors, $\mathcal{P}_{\mathcal{W}^2}^{(m, s_i)}(p)$, in terms of the Casimir invariants of the Poincaré algebra, the squared four-momentum, P^2 , and the squared Pauli-Lubanski vector, $\mathcal{W}^2(p)$ whose eigenvalues are given in the equation (2.15). Such can be done following the prescription given in [4] leading to,

$$\left[\mathcal{P}_{\mathcal{W}^2}^{(m, 1/2)}(p) \right]^{\alpha\beta} = \frac{P^2}{m^2} \left[\left(\frac{\mathcal{W}^2(p) - \epsilon_{3/2}}{\epsilon_{1/2} - \epsilon_{3/2}} \right) \right]^{\alpha\beta}, \quad (3.21)$$

$$\left[\mathcal{P}_{\mathcal{W}^2}^{(m, 3/2)}(p) \right]^{\alpha\beta} = \frac{P^2}{m^2} \left[\left(\frac{\mathcal{W}^2(p) - \epsilon_{1/2}}{\epsilon_{3/2} - \epsilon_{1/2}} \right) \right]^{\alpha\beta}. \quad (3.22)$$

Here, $\epsilon_{s_i} = -p^2 s_i(s_i + 1)$, with $s_1 = 1/2$, and $s_2 = 3/2$, is the $\mathcal{W}^2(p)$ eigenvalue corresponding to the mass- m and spin- s_i eigenstates of the operators P^2 and $\mathcal{W}^2(p)$. Defining now the tensor $\Gamma_{\mu\nu}^{(m, 1/2)}$ according to,

$$\left[\mathcal{P}_{\mathcal{W}^2}^{(m, 1/2)}(p) \right]_{\alpha\beta} = \frac{1}{m^2} \left[\Gamma_{\mu\nu}^{(m, 1/2)} \right]_{\alpha\beta} p^\mu p^\nu = \frac{p^2}{m^2} \left[\mathbb{P}^{(1/2)}(p) \right]_{\alpha\beta}, \quad (3.23)$$

where $\left[\mathbb{P}^{(1/2)}(p) \right]_{\alpha\beta}$ is the covariant spin-1/2 projector [15],

$$\left[\mathbb{P}^{(1/2)}(p) \right]_{\alpha\beta} = \frac{1}{3} \gamma_\alpha \gamma_\beta + \frac{1}{3p^2} (\not{p} \gamma_\alpha p_\beta + p_\alpha \gamma_\beta \not{p}), \quad (3.24)$$

one arrives at,

$$\left[\Gamma_{\mu\nu}^{(m, 1/2)} \right]_{\alpha\beta} = \frac{1}{6} (\sigma_{\alpha\mu} \sigma_{\beta\nu} + \sigma_{\alpha\nu} \sigma_{\beta\mu} + i \sigma_{\mu\nu} g_{\alpha\beta} + 5 g_{\alpha\nu} g_{\beta\mu} + g_{\alpha\mu} g_{\beta\nu}). \quad (3.25)$$

The spin $3/2^-$ projector, $\left[\mathcal{P}_{\mathcal{W}^2}^{(m, 3/2)}(p) \right]_{\alpha\beta}$ is then obtained as the difference between the unit operator in ψ_μ and $\left[\mathcal{P}_{\mathcal{W}^2}^{(m, 1/2)}(p) \right]_{\alpha\beta}$. It has been elaborated in detail in [4] and we limit ourselves to briefly present it in the Appendix II.

D. Full-space, and contracted wave equations for the $so(1, 3)$ irreducible spin-1/2⁺ and spin-1/2⁻ states in ψ_μ

So far we have been constructing the explicit degrees of freedom and have read off from their forms the wave equations and the conditions they obey, given in the Table 1. In the current subsection we focus on the inverse problem, namely, we shall be seeking for a general method of identifying the irreducible degrees of freedom in the four-vector-spinor and their wave equations, from which the explicit forms listed in the previous Section 2 can be obtained as solutions. Though we shall be keeping same notations as above, no use of the explicit expressions will be made. In order to find the equations of motion satisfied by the irreducible spin-1/2⁺ and spin-1/2⁻ spinors, the following eigenvalue problem needs to be solved,

$$\begin{aligned} \Pi_{(1/2, j_1) \oplus (j_1, 1/2)}(p) w_{\pm}^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) &= w_{\pm}^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right), \quad j_1 = 0, 1, \\ \Pi_{(1/2, j_1) \oplus (j_1, 1/2)}(p) &= \mathcal{P}_F^{(1/2, j_1)} \mathcal{P}_{\mathcal{W}^2}^{(m, 1/2)}(p) = \mathcal{P}_{\mathcal{W}^2}^{(m, 1/2)}(p) \mathcal{P}_F^{(1/2, j_1)}, \end{aligned} \quad (3.26)$$

with $\Pi_{(1/2, j_1) \oplus (j_1, 1/2)}(p)$, to be termed to as Lorentz- and Poincaré invariant projectors, being the products of the covariant-mass- m and spin-1/2 projector from Eqs. (3.21), on the one side, with the respective Lorentz projectors on $(1/2, 0) \oplus (0, 1/2)$, and $(1/2, 1) \oplus (1, 1/2)$ from Eqs. (3.8), and (3.9), on the other.

Though the projector used, strictly speaking, does not fix the parity of the states, i.e. it does not distinguish between low case plus and minus indexes, and also does not fix the polarizations, λ , we here nonetheless shall stick to the full notation, to be used through the paper, for the purpose of not unnecessarily changing notations. This with the justification, that the states of fixed parities and polarizations at any rate solve the equation (3.26) (with the Lorentz- and Poincaré- projectors from the respective eqs. (3.10)-(3.11), and (3.23)) as,

$$[\Gamma_{\mu\nu}^{(1/2,j_1)}]_{\alpha\beta} p^\mu p^\nu = m^2 [\mathcal{P}_F^{(1/2,j_1)}]_\alpha \gamma [\mathcal{P}_{W^2}^{(m,1/2)}(p)]_{\gamma\beta} = [\mathcal{P}_F^{(1/2,j_1)}]_\alpha \gamma [\Gamma_{\mu\nu}^{(m,1/2)}]_{\gamma\beta} p^\mu p^\nu, \quad j_1 = 1, \quad (3.27)$$

where $[\Gamma_{\mu\nu}^{(m,1/2)}]_{\gamma\beta}$ has been previously defined in (3.25). The relevant kinetic tensor, $[\Gamma_{\mu\nu}^{(1/2,j_1)}]_{\alpha\beta} p^\mu p^\nu$, defining the wave equations for both the spin- $1/2^+$ and $1/2^-$, and given in the latter equation (3.27), contains several anti-symmetric $[\partial_\mu, \partial_\nu]$ terms which are identically vanishing at the free particle level. In picking up some of those terms and dropping others, several equivalent non-interacting equations can be obtained, which however will become distinguishable upon gauging, only one of them for each sector, being the physical. We will make our choices as explained below and in terms of the following short hands of orthogonal (4×4) matrices in the Dirac spinor indexes,

$$[f^{(1/2,0)}(p)]^\alpha = \frac{1}{2m} \gamma^\alpha \not{p}, \quad (3.28)$$

$$[f^{(1/2,1)}(p)]^\alpha = \frac{2}{\sqrt{3}m} (p^\alpha - \frac{1}{4} \gamma^\alpha \not{p}), \quad (3.29)$$

orthonormalized on the mass-shell according to,

$$[\bar{f}^{(1/2,j)}(p)]^\alpha [f^{(1/2,j')}(p)]_\alpha = \delta_{jj'} \frac{p^2}{m^2}, \quad (3.30)$$

with $[\bar{f}^{(1/2,j_1)}(p)]^\alpha = \gamma^0 ([f^{(1/2,j_1)}(p)]^\alpha)^\dagger \gamma^0$. The $[f^{(1/2,j_1)}(p)]^\alpha$ matrices bring the great advantage to bi-linearize the kinetic terms of the equations of motion (3.26) as,

$$(m^2 [f^{(1/2,j_1)}(p)]_\alpha [\bar{f}^{(1/2,j_1)}(p)]_\beta - m^2 g_{\alpha\beta}) \left[w_\pm^{(1/2,j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\beta = 0. \quad (3.31)$$

Factorizing the momenta, the equation (3.31) translates into,

$$([\tilde{\Gamma}_{\mu\nu}^{(1/2,j_1)}]_{\alpha\beta} p^\mu p^\nu - m^2 g_{\alpha\beta}) \left[w_\pm^{(1/2,j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\beta = 0, \quad j_1 = 0, 1, \quad (3.32)$$

where new tensor $[\tilde{\Gamma}_{\mu\nu}^{(1/2,j_1)}]_{\alpha\beta}$ will be the one to be used systematically in the following.

Spin- $\frac{1}{2}^+$:

The spin- $1/2^+$ wave equation following from (3.28) and (3.31) reads,

$$\left(\frac{1}{4} (\gamma_\alpha \not{p} \not{p} \gamma_\beta)_{ab} - m^2 \delta_{ab} g_{\alpha\beta} \right) \left[w_\pm^{(1/2,0)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^{\beta b} = 0, \quad (3.33)$$

where we as an exception and temporarily brought back the Dirac indexes, a, b for the sake of rising the transparency of the discussion on the solutions to follow. Notice that this is a genuine quadratic and relativistic fermion equation, not a scalar one. The information on the dimensionality of the four-vector-spinor space has fully been encoded by the Lorentz indexes of the gamma matrices and their Dirac indexes.

In order to find out the solutions, we notice that the Dirac index of $\left[w_\pm^{(1/2,0)}(\mathbf{p}, 1/2, \lambda) \right]_{\alpha b}^{\alpha b}$ has to be carried by a spinor, $[u_\pm(\mathbf{p}, \lambda)]_b$, while the Lorentz index can be carried either by $p_\alpha \left(\frac{\not{p}}{m} \right)^{\alpha b}_{n_1}$, or by

$\gamma_\alpha \left(\frac{p}{m}\right)^{n_2}$, for $n_1 \geq 0$, and $n_2 \geq 0$. The first solution is ruled out for not being an $\mathcal{P}_F^{(1/2,0)}$ eigenstate, while the second satisfies (3.33), and one can chose the lowest $n_2 = 0$ value. In this way, the explicit form previously found in (3.16) is recovered modulo a normalization constant. In consequence, the four-vector spinors describing the irreducible Dirac sector are such that their contractions with the γ -matrices behave as solutions to the Dirac equation, meaning,

$$\begin{aligned} \gamma_{ab}^\alpha \left[w_\pm^{(1/2,0)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_{\alpha b} &\sim [u_\pm(\mathbf{p}, \lambda)]_a, \\ (\not{p} \mp m) \left[\gamma_{ab}^\alpha \left[w_\pm^{(1/2,0)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_{\alpha b} \right] &= 0. \end{aligned} \quad (3.34)$$

Finally, contraction of (3.33) by γ^α from the left yields,

$$(\not{p} \delta_{ab} - m^2 \delta_{ab}) \left[\gamma \cdot w_\pm^{(1/2,0)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_b = 0. \quad (3.35)$$

Taking into account that,

$$\not{p} \not{p} = p^2 - i \frac{\sigma^{\mu\nu}}{2} [p_\mu, p_\nu], \quad (3.36)$$

one arrives at,

$$\left(p^2 - i \frac{g(1/2^+)}{4} \sigma^{\mu\nu} [p_\mu, p_\nu] - m^2 \right) \left[\gamma \cdot w_\pm^{(1/2,0)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right] = 0, \quad (3.37)$$

with $g(1/2^+) = 2$, to be independently confirmed below. The equation (3.34) shows that $\left[\gamma \cdot w_\pm^{(1/2,0)}(\mathbf{p}, 1/2, \lambda) \right]$ satisfies the Dirac equation and behaves as a free spin-1/2⁺ Dirac particle, while (3.35) confirms that each spinor component satisfies the free Klein-Gordon equation, as it should be for a free particle on on mass-shell.

Spin- $\frac{1}{2}^-$:

The emerging wave equation relevant for this case can be cast as,

$$\left[\frac{4}{3} \left(p_\alpha \delta_{ab} - \frac{1}{4} [\gamma_\alpha]_{ab} \not{p} \right) \left(p_\beta \delta_{ab} - \frac{1}{4} \not{p} [\gamma_\beta]_{ab} \right) - m^2 \delta_{ab} g_{\alpha\beta} \right] \left[w_\mp^{(1/2,1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^{\beta b} = 0. \quad (3.38)$$

The latter incorporates the following property, revealed through its contraction by γ^α ,

$$-m^2 [\gamma_\beta]_{ab} \left[w_\pm^{(1/2,1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^{\beta b} = 0. \quad (3.39)$$

It needs to be emphasized that within the second order formalism advocated here, the latter equation does not play the role of an auxiliary condition. The principle equation (3.38) is free from auxiliary conditions, and the equation (3.39) reflects only one of its properties. The meaning of (3.39) is that $w_\pm^{(1/2,1)}(\mathbf{p}, 1/2, \lambda)$ does not have any projection on $(1/2, 0) \oplus (0, 1/2)$, as it should be, and in accord with the $so(1, 3)$ reduction of the Rarita-Schwinger space established in (2.1). The Dirac index, b , of the $\left[w_\pm^{(1/2,1)}(\mathbf{p}, 1/2, \lambda) \right]_{\beta b}$ spinor has to be carried by a spinor, while the Lorentz index, β , can be carried equally well by the four momentum, p_β , on the one side, and by the Dirac matrices, γ_β , on the other, ending up by a linear combination of them,

$$\left[w_\pm^{(1/2,1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_{\beta b} = c_1 p_\beta \delta_{ab} [u_\pm(\mathbf{p}, \lambda)]_b + c_2 [\gamma_\beta]_{ab} [u_\pm(\mathbf{p}, \lambda)]_b, \quad c_2 = -\frac{1}{4} c_1. \quad (3.40)$$

The coefficients are determined from the requirement that the states also diagonalize the Lorentz projector, $\mathcal{P}_F^{(1/2,1)}$, in combination with their normalizations. In this way, the explicit form previously known by construction from (3.17), can be recovered, yielding,

$$\begin{aligned} \left[p \cdot w_{\pm}^{(1/2,1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_a &\sim [u_{\mp}(\mathbf{p}, \lambda)]_a, \\ (\not{p} \pm m) \left[p \cdot w_{\pm}^{(1/2,1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]_a &= 0. \end{aligned} \quad (3.41)$$

In this fashion, one confirms that also $p \cdot w_{\pm}^{(1/2,1)}(\mathbf{p}, 1/2, \lambda)$ is described in terms of an ordinary Dirac equation and therefore behaves as a free spin-1/2⁻.

Finally, contraction of (3.38) by p^α from the left yields,

$$\left[\frac{4}{3} \left(p^2 - \frac{1}{4} \not{p} \not{p} \right) - m^2 \right] \left[p \cdot w_{\pm}^{(1/2,1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right] - \frac{1}{3} \left(p^2 - \frac{1}{4} \not{p} \not{p} \right) \not{p} \left[\gamma \cdot w_{\pm}^{(1/2,1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right] = 0 \quad (3.42)$$

where the last term can be ignored by virtue of the property in (3.39). In effect, the wave equation relevant for the case under consideration assumes the form,

$$\left(p^2 - \frac{1}{4} \not{p} \not{p} - \frac{3}{4} m^2 \right) \left[p \cdot w_{\pm}^{(1/2,1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right] = 0. \quad (3.43)$$

Taking into account (3.36), and factorizing 3/4, the equation (3.43) equivalently rewrites to,

$$\left(p^2 - i \frac{g(1/2^-)}{4} \sigma^{\mu\nu} [p_\mu, p_\nu] - m^2 \right) \left[p \cdot w_{\pm}^{(1/2,1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right], \quad (3.44)$$

with $g(1/2^-) = -2/3$, to be independently confirmed below. As a reminder, at the free particle level, the presence of the $[p^\mu, p^\nu]$ term is irrelevant and the equation is a pure scalar one, in which the information of the dimensionality of the $so(1,3)$ representation space is completely lost. However, this changes upon gauging, when the term becomes viable by virtue of

$$[\pi^\mu, \pi^\nu] = -ieF^{\mu\nu}. \quad (3.45)$$

In effect, the gauged equation is no longer scalar but becomes $(1/2, 0) \oplus (0, 1/2)$ representation specific, i.e. of four spinorial dimensions.

In the following, when considering the gauging procedure, preference to the equations (3.37) and (3.44) over (3.35) and (3.43) will be given. For free particles instead, the equations (3.33) and (3.38) will be systematically used. The free $so(1,3)$ irreducible spin-1/2⁺ and spin-1/2⁻ degrees of freedom in the four-vector–spinor, together with the respective free particle wave equations, are listed in the Table 2.

Finally, the propagators of the two spin-1/2 sectors transforming according to different j_1 -labels are obtained as the inverse to the respective equation operators as,

$$[S^{(1/2, j_1)}(p)]_{\alpha\beta} = \left([\Gamma_{\mu\nu}^{(1/2, j_1)}]_{\alpha\beta} p^\mu p^\nu - m^2 g_{\alpha\beta} \right)^{-1}, \quad (3.46)$$

and are given by

$$[S^{(1/2, j_1)}(p)]_{\alpha\beta} = \frac{[\Delta^{(1/2, j_1)}(p)]_{\alpha\beta}}{p^2 - m^2 + i\epsilon}, \quad (3.47)$$

where

$$[\Delta^{(1/2, 0)}(p)]_{\alpha\beta} = \frac{1}{m^2} \left(\frac{1}{4} p^2 \gamma_\alpha \gamma_\beta + (m^2 - p^2) g_{\alpha\beta} \right), \quad (3.48)$$

$$[\Delta^{(1/2, 1)}(p)]_{\alpha\beta} = \frac{1}{m^2} \left[\frac{4}{3} \left(p_\alpha - \frac{1}{4} \gamma_\alpha \not{p} \right) \left(p_\beta - \frac{1}{4} \not{p} \gamma_\beta \right) + (m^2 - p^2) g_{\alpha\beta} \right]. \quad (3.49)$$

$so(1,3)$ irreducible free spin-1/2 degrees of freedom	Free particle wave equations
<p><u>spin-1/2⁺ $\in (1/2, 0) \oplus (0, 1/2)$:</u></p> $\left[w_{\pm}^{(1/2,0)}(\mathbf{p}, \frac{1}{2}, \lambda) \right]^{\alpha} = \frac{1}{2m} \gamma^{\alpha} \not{p} u_{\pm}(\mathbf{p}, \lambda)$	<p><u>In the full space:</u></p> $\left(\frac{1}{4} (\gamma_{\alpha} \not{p} \gamma_{\beta})_{ab} - m^2 \delta_{ab} g_{\alpha\beta} \right) \left[w_{\pm}^{(1/2,0)}(\mathbf{p}, \frac{1}{2}, \lambda) \right]^{\beta b} = 0 \quad \text{eq. (3.33)}$ <p><u>Contracted:</u></p> $\left(p^2 - i \frac{g_{(1/2^+)}}{4} \sigma^{\mu\nu} [p_{\mu}, p_{\nu}] - m^2 \right) \left[\gamma \cdot w_{\pm}^{(1/2,0)}(\mathbf{p}, \frac{1}{2}, \lambda) \right] = 0$ <p>eq. (3.37)</p>
<p><u>spin-1/2⁻ $\in (1/2, 1) \oplus (1, 1/2)$:</u></p> $\left[w_{\pm}^{(1/2,1)}(\mathbf{p}, \frac{1}{2}, \lambda) \right]^{\alpha} = \frac{2}{\sqrt{3}m} (p^{\alpha} - \frac{1}{4} \gamma^{\alpha} \not{p}) u_{\pm}(\mathbf{p}, \lambda)$	<p><u>In the full space:</u></p> $\left[\frac{4}{3} (p_{\alpha} \delta_{ab} - \frac{1}{4} [\gamma_{\alpha}]_{ab} \not{p}) (p_{\beta} \delta_{ab} - \frac{1}{4} \not{p} [\gamma_{\beta}]_{ab}) - m^2 \delta_{ab} g_{\alpha\beta} \right]$ $\times \left[w_{\mp}^{(1/2,1)}(\mathbf{p}, \frac{1}{2}, \lambda) \right]^{\beta b} = 0 \quad \text{eq. (3.38)}$ <p><u>Contracted:</u></p> $\left(p^2 - i \frac{g_{(1/2^-)}}{4} \sigma^{\mu\nu} [p_{\mu}, p_{\nu}] - m^2 \right) \left[p \cdot w_{\pm}^{(1/2,1)}(\mathbf{p}, \frac{1}{2}, \lambda) \right] = 0$ <p>eq. (3.44)</p>

TABLE II: Summary of the $so(1,3)$ irreducible degrees of freedom spanning the four-vector-spinor space (left column), and their description by second order wave equations (right column) free from auxiliary conditions. We list for each one of the spin-1/2⁺ $\in (1/2, 0) \oplus (0, 1/2)$, and spin-1/2⁻ $\in (1/2, 1) \oplus (1, 1/2)$ sectors two types of wave equations, the first referring to the full 16 dimensional four-vector-spinor space, and the second, obtained from the previous one by either γ^{α} , or, p^{α} , contraction, referring to four spinorial degrees of freedom. In section 5 below we write down the Lagrangians associated with all these equations, their gauged forms being given then in Section 6. In section 6 we show that Compton scattering off spin-1/2⁻ does not distinguish between the full-space- and the contracted Lagrangians. From that we conclude that the covariant spin-1/2⁻-spin-3/2⁻ separation performed at the free particle level is respected by the gauging procedure and that our approach continues describing the gauged four spin-1/2⁻ by spinorial degrees of freedom, as it should be. On this basis we conclude that the spin-1/2⁻ particle under discussion effectively behaves as a true “quadratic” fermion, as the contracted gauged quadratic equation does not allow for a bi-linearization because of $g_{(1/2^-)} \neq 2$.

In terms of the Lorentz- and Poincaré- projectors from the respective eqs. (3.10)-(3.11), and (3.23), the latter quantities are equivalent to,

$$[\Delta^{(1/2,j_1)}(p)]_{\alpha\beta} = \frac{p^2}{m^2} [\mathcal{P}_F^{(1/2,j_1)}]_{\alpha}{}^{\gamma} [\mathbb{P}^{(1/2)}(p)]_{\gamma\beta} + \frac{(m^2 - p^2)}{m^2} g_{\alpha\beta}, \quad (3.50)$$

while in terms of the $[f^{(1/2,j_1)}(p)]^{\alpha}$ matrices one has,

$$[\Delta^{(1/2,j_1)}(p)]_{\alpha\beta} = [f^{(1/2,j_1)}(p)]_{\alpha}{}^{\gamma} [\bar{f}^{(1/2,j_1)}(p)]_{\gamma\beta} + \frac{(m^2 - p^2)}{m^2} g_{\alpha\beta}. \quad (3.51)$$

The conclusion is that while the 16 dimensional $so(1, 3)$ irreducible four-vector spinors describing its spin-1/2⁺ and spin-1/2⁻ sectors do not satisfy the Dirac equation, their contractions by γ^α and p^α down to four spinorial degrees of freedom each, do. Within this context the question arises, as to what extend the contracted four-vector spinors are eligible for the description of the two spin-1/2⁺ and spin-1/2⁻ particles under discussion. For the time being, and at the free particle level, no distinction can be made between these degrees of freedom, namely, between $[\gamma \cdot w_\pm^{(1/2,0)}(\mathbf{p}, \frac{1}{2}, \lambda)]$, and $[p \cdot w_\mp^{(1/2,1)}(\mathbf{p}, \frac{1}{2}, \lambda)]$, in the respective equations (3.34) and (3.41), because they both satisfy the free Dirac equation, and behave as spin-1/2⁺, and spin-1/2⁻, respectively. This situation is to change upon introducing interactions, an issue handled in the subsequent section.

IV. THE ELECTROMAGNETICALLY GAUGED CONTRACTED WAVE EQUATIONS

In the current and the subsequent sections we consider the couplings of the free spin-1/2⁺ and spin-1/2⁻ particle equations, (3.33), (3.37), and (3.38), (3.44) to the electromagnetic field, confining to minimal gauging in the sense that the coupling is defined through changing ordinary by covariant derivatives according to,

$$\partial^\mu \longrightarrow D^\mu = \partial^\mu + ieA^\mu, \quad (4.1)$$

where e is the electric charge of the particle. In order to obtain the gauged equations in the full space, we first write (3.32) in position space for a plane wave of the type $[\psi_\pm^{(1/2,j_1)}]^\alpha(x) \longrightarrow [w_\pm^{(1/2,j_1)}(\mathbf{p}, 1/2, \lambda)]^\alpha e^{\mp i x \cdot p}$, as

$$\left([\tilde{\Gamma}_{\mu\nu}^{(1/2,j_1)}]_{\alpha\beta} \partial^\mu \partial^\nu + m^2 g_{\alpha\beta} \right) [\psi_\pm^{(1/2,j_1)}(x)]^\beta = 0. \quad (4.2)$$

This is in reality a (16×16) dimensional matrix equation for the 16-component vector-spinor $[\psi_\pm^{(1/2,j_1)}(x)]^\beta$. However, we have shown in the momentum space equations (3.35) and (3.42) above that for spin-1/2 in ψ_μ the contractions of the latter be it by γ^α , or p^α , are four-dimensional spinors, which translates to position space as, $[\gamma \cdot \psi_\pm^{(1/2,0)}(x)]$, and $[i\partial \cdot \psi_\pm^{(1/2,0)}(x)]$, respectively. Below we gauge these two schemes and reveal their equivalence.

1. Gauging the contracted wave equation for the $so(1,3)$ irreducible $(1/2, 0) \oplus (0, 1/2)$ sector

Replacing now in (3.37) everywhere the ordinary by the covariant derivatives, $\partial^\mu \longrightarrow D^\mu$, gives

$$\left(D^2 + g_{(1/2+)} \frac{e}{4} \sigma^{\mu\nu} F_{\mu\nu} + m^2 \right) [\gamma \cdot \psi_\pm^{(1/2,0)}(x)] = 0. \quad (4.3)$$

For $g_{(1/2+)} = 2$ the gauged $[\gamma \cdot \psi_\pm^{(1/2,0)}(x)]$ spin-1/2⁺ solution in $(1/2, 0) \oplus (0, 1/2)$, would be proportional to the gauged Dirac spinor,

$$\not{D} [\gamma \cdot \psi_\pm^{(1/2,0)}(x)] = -im [\gamma \cdot \psi_\pm^{(1/2,0)}(x)] \implies [\gamma \cdot \psi_\pm^{(1/2,0)}(x)] \sim \psi_\pm^D(x). \quad (4.4)$$

In other words, both $[\gamma \cdot \psi_\pm^{(1/2,0)}(x)]$, and $\psi_\pm^D(x)$ now solve the gauged Dirac equation [16] and describe same spin-1/2⁺ particle of a gyromagnetic factor $g_{(1/2+)} = 2$.

2. Gauging the contracted wave equation for spin-1/2⁻ in the $so(1,3)$ irreducible $(1/2, 1) \oplus (1, 1/2)$ sector

Along same line as in the previous subsection gauging (3.44) amounts to,

$$\left(D^2 + \frac{g_{(1/2-)}}{4} e \sigma^{\mu\nu} F_{\mu\nu} + m^2 \right) [D \cdot \psi_\mp^{(1/2,1)}(x)] = 0, \quad (4.5)$$

where $g_{(1/2-)} = -2/3$ takes the place of the physical gyromagnetic factor of the spin- $1/2^-$ particle under consideration. Remarkably, both the equations (4.3) and (4.5) are of the art of the equation (1.11) following from the Lagrangian in (1.12). This circumstance will allow us to decide whether $[D \cdot \psi_{\mp}^{(1/2,1)}(x)]$ behaves as a Dirac-, or, as a quadratic fermion, in depending on whether or not its $g_{(1/2-)}$ value happens to be confirmed by the magnetic dipole moment calculated from the electromagnetic current in the full 16 dimensional four-vector spinor space. In the next section we shall present the Lagrangians associated with the gauged equations within the full space, obtain the electromagnetic currents, and calculate the magnetic dipole moments for both the spin- $1/2$ sectors under consideration. This will allow us to figure out the true nature of spin- $1/2^-$.

V. THE GAUGED LAGRANGIANS OF THE TWO SPIN- $1/2$ SECTORS OF THE FOUR-VECTOR FIELD WITHIN THE FULL SPACE

For practical calculations it is advantageous to have at our disposal gauged Lagrangians, from which one can extract the Feynman rules of the theory. The Lagrangians for positive and negative parity states differ only by an overall sign that reflects the relative sign in the normalizations of the opposite parity states. In the following we shall only deal with Lagrangians written in terms of the positive parity states. The free equations of motion (3.33) and (3.43) relate to Lagrangians in the standard way as Euler-Lagrange equations. For second order equations of motion the corresponding Lagrangians are of the form,

$$\mathcal{L}_{\text{free}}^{(1/2,j_1)}(x) = (\partial^\mu [\bar{\psi}_+^{(1/2,j_1)}(x)]^\alpha) [\tilde{\Gamma}_{\mu\nu}^{(1/2,j_1)}]_{\alpha\beta} \partial^\nu [\psi_+^{(1/2,j_1)}(x)]^\beta - m^2 [\bar{\psi}_+^{(1/2,j_1)}(x)]^\alpha [\psi_+^{(1/2,j_1)}(x)]_\alpha, \quad (5.1)$$

and the gauged Lagrangians are obtained as usually by the replacement of the ordinary- by covariant derivatives,

$$\mathcal{L}^{(1/2,j_1)}(x) = (D^{\mu*} [\bar{\psi}_+^{(1/2,j_1)}(x)]^\alpha) [\tilde{\Gamma}_{\mu\nu}^{(1/2,j_1)}]_{\alpha\beta} D^\nu [\psi_+^{(1/2,j_1)}(x)]^\beta - m^2 [\bar{\psi}_+^{(1/2,j_1)}(x)]^\alpha [\psi_+^{(1/2,j_1)}(x)]_\alpha. \quad (5.2)$$

After the use of (3.33), the Lagrangian with $j_1 = 0$, which corresponds to the sector $(1/2, 0) \oplus (0, 1/2)$, assumes the following explicit form,

$$\mathcal{L}^{(1/2,0)}(x) = \frac{1}{4} (D^{\mu*} [\bar{\psi}_+^{(1/2,0)}(x)]^\alpha) \gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\beta [\psi_+^{(1/2,0)}(x)]^\beta - m^2 [\bar{\psi}_+^{(1/2,0)}(x)]^\alpha D^\mu [\psi_+^{(1/2,0)}(x)]_\alpha. \quad (5.3)$$

while the $j_1 = 1$ Lagrangian, which is the one relevant for the $(1/2, 1) \oplus (1, 1/2)$ sector reads,

$$\begin{aligned} \mathcal{L}^{(1/2,1)}(x) &= \frac{4}{3} (D^{\mu*} [\bar{\psi}_+^{(1/2,1)}(x)]^\alpha) \left(g_{\alpha\mu} - \frac{1}{4} \gamma_\alpha \gamma_\mu \right) \left(g_{\nu\beta} - \frac{1}{4} \gamma_\nu \gamma_\beta \right) [\psi_+^{(1/2,1)}(x)]^\beta \\ &\quad - m^2 [\bar{\psi}_+^{(1/2,1)}(x)]^\alpha [\psi_+^{(1/2,1)}(x)]_\alpha, \end{aligned} \quad (5.4)$$

where we made use of (3.38). Now $\mathcal{L}^{(1/2,j_1)}(x)$ can be decomposed into free and interaction Lagrangians as,

$$\mathcal{L}^{(1/2,j_1)}(x) = \mathcal{L}_{\text{free}}^{(1/2,j_1)}(x) + \mathcal{L}_{\text{int}}^{(1/2,j_1)}(x), \quad (5.5)$$

$$\mathcal{L}_{\text{int}}^{(1/2,j_1)}(x) = -j_\mu^{(1/2,j_1)} A^\mu(x) + k_{\mu\nu}^{(1/2,j_1)}(x) A^\mu(x) A^\nu(x). \quad (5.6)$$

Here, $j_\mu^{(1/2,j_1)}(x)$ is the electromagnetic current density, while $k_{\mu\nu}^{(1/2,j_1)}(x)$ defines the structure of the two-photon coupling. Back to momentum space, and for the positive parity states $[w_+^{(1/2,j_1)}(\mathbf{p}, 1/2, \lambda)]^\beta$ we find,

$$j_\mu^{(1/2,j_1)}(\mathbf{p}, \mathbf{p}', \lambda, \lambda') = e \left[\bar{w}_+^{(1/2,j_1)} \left(\mathbf{p}', \frac{1}{2}, \lambda' \right) \right]^\alpha [\mathcal{V}_\mu^{(1/2,j_1)}(p', p)]_{\alpha\beta} \left[w_+^{(1/2,j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\beta, \quad (5.7)$$

$$k_{\mu\nu}^{(1/2,j_1)}(\mathbf{p}, \mathbf{p}', \lambda, \lambda') = e^2 \left[\bar{w}_+^{(1/2,j_1)} \left(\mathbf{p}', \frac{1}{2}, \lambda' \right) \right]^\alpha [\mathcal{C}_{\mu\nu}^{(1/2,j_1)}]_{\alpha\beta} \left[w_+^{(1/2,j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\beta, \quad (5.8)$$

Fig. 1: Feynman rule for the propagators of particles in the $(1/2, j_1) \oplus (j_1, 1/2)$ sector of the four-vector-spinor. They are obtained as the inverse of the equations of motion, the explicit form of $S^{(1/2, j_1)}(p)$ for each j_1 -value is given in (3.46)-(3.49).

Fig. 2: Feynman rule for the one-photon vertex with $(1/2, j_1) \oplus (j_1, 1/2)$ particles. It comes from the interaction Lagrangian (5.6), for the specific definition of $\mathcal{V}_\mu^{(1/2, j_1)}(p', p)$ corresponding to each j_1 -value see (5.9).

Fig. 3: Diagram for the two-photons Feynman rule corresponding to the interaction Lagrangian (5.6), the explicit form of the $\mathcal{C}_{\mu\nu}^{(1/2, j_1)}$ vertex is given in (5.10).

with

$$[\mathcal{V}_\mu^{(1/2, j_1)}(p', p)]_{\alpha\beta} = [\tilde{\Gamma}_{\nu\mu}^{(1/2, j_1)}]_{\alpha\beta} p'^\nu + [\tilde{\Gamma}_{\mu\nu}^{(1/2, j_1)}]_{\alpha\beta} p'^\nu, \quad (5.9)$$

$$[\mathcal{C}_{\mu\nu}^{(1/2, j_1)}]_{\alpha\beta} = \frac{1}{2} \left([\tilde{\Gamma}_{\mu\nu}^{(1/2, j_1)}]_{\alpha\beta} + [\tilde{\Gamma}_{\nu\mu}^{(1/2, j_1)}]_{\alpha\beta} \right). \quad (5.10)$$

Here, $[\mathcal{V}_\mu^{(1/2, j_1)}(p', p)]_{\alpha\beta}$ and $[\mathcal{C}_{\mu\nu}^{(1/2, j_1)}]_{\alpha\beta}$ are the respective one- and two-photon vertexes which, together with the propagators (3.47), determine the Feynman rules. The latter are depicted on the Figs. 1, 2, 3.

In particular, $[\mathcal{V}_\mu^{(1/2, j_1)}(p', p)]_{\alpha\beta}$ obeys the Ward-Takahashi identity,

$$(p' - p)^\mu [\mathcal{V}_\mu^{(1/2, j_1)}(p', p)]_{\alpha\beta} = [S^{(1/2, j_1)}(p')]_{\alpha\beta}^{-1} - [S^{(1/2, j_1)}(p)]_{\alpha\beta}^{-1}, \quad (5.11)$$

where $[S^{(1/2, j_1)}(p)]_{\alpha\beta}$ are the propagators in (3.47). This relationship implies gauge invariance of the amplitudes which define the Compton scattering process. However, before evaluating this very process it is instructive to calculate the values of the magnetic dipole moments of the particles under consideration, as they appear prescribed by the currents in (5.7).

A. Magnetic dipole moments

We now rewrite the momentum space currents in (5.7) in terms of free Dirac u -spinors as,

$$j_\mu^{(1/2, j_1)}(\mathbf{p}, \mathbf{p}', \lambda, \lambda') = e \bar{u}_+(\mathbf{p}', \lambda') \tilde{\mathcal{V}}_\mu^{(1/2, j_1)}(p', p) u_+(\mathbf{p}, \lambda). \quad (5.12)$$

Here we have used the equations (3.16)-(3.17) to come to,

$$\left[w_\pm^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\alpha = [f^{(1/2, j_1)}(p)]^\alpha u_\pm(\mathbf{p}, \lambda), \quad (5.13)$$

and on the basis of this relationship defined the new vertex, $\tilde{\mathcal{V}}_\mu^{(1/2,j_1)}(p', p)$, as

$$\tilde{\mathcal{V}}_\mu^{(1/2,j_1)}(p', p) = [\bar{f}^{(1/2,j_1)}(p')]^\alpha [\mathcal{V}_\mu^{(1/2,j_1)}(p', p)]_{\alpha\beta} [f^{(1/2,j_1)}(p)]^\beta, \quad j_1 = 0, 1. \quad (5.14)$$

Incorporation of the mass-shell condition, and $\not{p} u_+(\mathbf{p}, \lambda) = m u_+(\mathbf{p}, \lambda)$, amounts to,

$$j_\mu^{(1/2,0)}(\mathbf{p}', \mathbf{p}, \lambda, \lambda') = e \bar{u}_+(\mathbf{p}', \lambda') (2m \gamma_\mu) u_+(\mathbf{p}, \lambda), \quad (5.15)$$

$$j_\mu^{(1/2,1)}(\mathbf{p}', \mathbf{p}, \lambda, \lambda') = e \bar{u}_+(\mathbf{p}', \lambda') \left(\frac{4}{3} (p' + p)_\mu - \frac{2m}{3} \gamma_\mu \right) u_+(\mathbf{p}, \lambda). \quad (5.16)$$

We now perform the Gordon decomposition of the first current, yielding,

$$2m e \bar{u}_+(\mathbf{p}', \lambda') \gamma_\mu u_+(\mathbf{p}, \lambda) = e \bar{u}_+(\mathbf{p}', \lambda') [(p' + p)_\mu + 2i M_{\mu\nu}^S (p' - p)^\nu] u_+(\mathbf{p}, \lambda). \quad (5.17)$$

Here $M_{\mu\nu}^S$ are the elements of the Lorentz algebra in $(1/2, 0) \oplus (0, 1/2)$ in (2.10), while the factor 2 in front of them stands for the gyromagnetic ratio. Therefore, one immediately notices that for $j_1 = 0$, the textbook Dirac current with $g_{(1/2+)} = 2$ is recovered. Similarly, the second current is handled. As a result, both currents in (5.15) and (5.16) are equally shaped after,

$$j_\mu^{(1/2,j_1)}(\mathbf{p}', \mathbf{p}, \lambda, \lambda') = e \bar{u}_+(\mathbf{p}', \lambda') [(p' + p)_\mu + i g_{s_i} M_{\mu\nu}^S (p' - p)^\nu] u_+(\mathbf{p}, \lambda), \quad (5.18)$$

$$s_1 = 1/2^+, \quad s_2 = 1/2^-,$$

with

$$g_{(1/2+)} = 2, \quad (5.19)$$

$$g_{(1/2-)} = -\frac{2}{3}. \quad (5.20)$$

The equations (5.19) and (5.20) show that the electromagnetic currents for particles transforming in $(1/2, j_1) \oplus (j_1, 1/2)$ are characterized by different magnetic dipole moments for the two different $j_1 = 0, 1$ values in $(1/2, j_1) \oplus (j_1, 1/2)$. The gauged Lagrangian corresponding to the combined Lorentz- and Poincaré invariant projector, that describes particles of charge e transforming in $(1/2, 0) \oplus (0, 1/2)$ predicts the following magnetic moment,

$$\mu_{(1/2+)}(\lambda) = 2 \frac{\lambda e}{2m}. \quad (5.21)$$

The latter coincides with the standard value for a Dirac particle of polarization λ . Instead, the Lagrangian of same type predicts for particles of charge e transforming in $(1/2, 1) \oplus (1, 1/2)$ a magnetic dipole moment of

$$\mu_{(1/2-)}(\lambda) = -\frac{2}{3} \frac{\lambda e}{2m}. \quad (5.22)$$

Although these g_s values coincide with those in (3.37) and (3.44), it is precipitate to claim equivalence between the effective spinorial degrees of freedom solving the contracted equations, and the genuine four-vector-spinor degrees of freedom describing the two spin-1/2 sectors under consideration. The reason is that an electromagnetic process is not entirely determined by the electromagnetic multipole moments of the particles, which by definition are associated with the on-shell states, it is determined by the complete gauged Lagrangian. The mere confirmation of the electromagnetic multipole moments by a theory is not sufficient to claim its credibility. This because different Lagrangians can predict equal multipole moments.[17],[18]. The more profound test for the predictive power of Lagrangians concerns processes involving off-shell states. One such process, the Compton scattering, is the subject of the next section.

VI. COMPTON SCATTERING OFF SPIN-1/2 IN $(1/2, j_1) \oplus (j_1, 1/2)$ WITH $j_1 = 0, 1$

A. The calculation within the full space

The construction of the Compton scattering amplitudes from the Feynman rules (shown in the Figs. 1, 2, and 3) for each j_1 - value, is standard [7] and reads,

$$\mathcal{M}^{(1/2,j_1)} = \mathcal{M}_A^{(1/2,j_1)} + \mathcal{M}_B^{(1/2,j_1)} + \mathcal{M}_C^{(1/2,j_1)}, \quad (6.1)$$

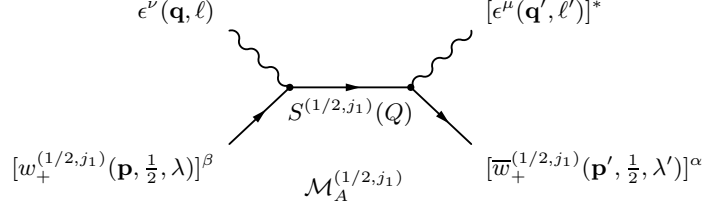


Fig. 4: Diagram for the direct-scattering contribution (6.2) to the Compton scattering amplitude (6.1).

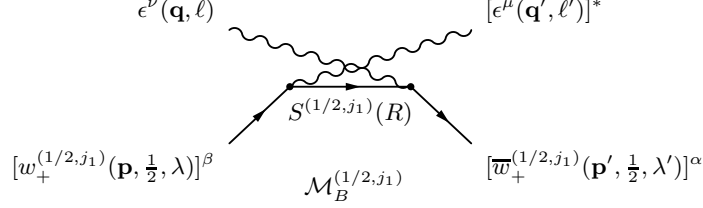


Fig. 5: Diagram for the exchange-scattering contribution (6.3) to the Compton scattering amplitude (6.1).

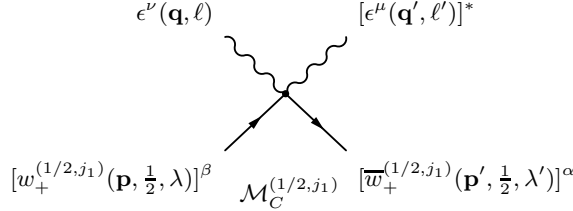


Fig. 6: Diagram for the point-scattering contribution (6.4) to the Compton scattering amplitude (6.1).

where $\mathcal{M}_A^{(1/2, j_1)}$, $\mathcal{M}_B^{(1/2, j_1)}$, $\mathcal{M}_C^{(1/2, j_1)}$ correspond to the amplitudes for direct, exchange and contact scatterings, respectively. In the following we use p and p' to denote the momentum of the incident and scattered spin-1/2 particles respectively, while q and q' stand in their turn for the momenta of the incident and scattered photons. In effect, we find,

$$i\mathcal{M}_A^{(1/2, j_1)} = e^2 \left[\bar{w}_+^{(1/2, j_1)} \left(\mathbf{p}', \frac{1}{2}, \lambda' \right) \right]^\alpha [U_{\mu\nu}^{(1/2, j_1)}(p', Q, p)]_{\alpha\beta} \left[w_+^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\beta \times [\epsilon^\mu(\mathbf{q}', \ell')]^* \epsilon^\nu(\mathbf{q}, \ell), \quad (6.2)$$

$$i\mathcal{M}_B^{(1/2, j_1)} = e^2 \left[\bar{w}_+^{(1/2, j_1)} \left(\mathbf{p}', \frac{1}{2}, \lambda' \right) \right]^\alpha [U_{\nu\mu}^{(1/2, j_1)}(p', R, p)]_{\alpha\beta} \left[w_+^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\beta \times [\epsilon^\mu(\mathbf{q}', \ell')]^* \epsilon^\nu(\mathbf{q}, \ell), \quad (6.3)$$

$$i\mathcal{M}_C^{(1/2, j_1)} = -e^2 \left[\bar{w}_+^{(1/2, j_1)} \left(\mathbf{p}', \frac{1}{2}, \lambda' \right) \right]^\alpha [\mathcal{C}_{\mu\nu}^{(1/2, 0)} + \mathcal{C}_{\nu\mu}^{(1/2, 0)}]_{\alpha\beta} \left[w_+^{(1/2, j_1)} \left(\mathbf{p}, \frac{1}{2}, \lambda \right) \right]^\beta \times [\epsilon^\mu(\mathbf{q}', \ell')]^* \epsilon^\nu(\mathbf{q}, \ell), \quad (6.4)$$

where $Q = p + q = p' + q'$ and $R = p' - q = p - q'$ stand for the momentum of the intermediate states, while

$$[U_{\mu\nu}^{(1/2, j_1)}(p', Q, p)]_{\alpha\beta} = [\mathcal{V}_\mu^{(1/2, j_1)}(p', Q)]_{\alpha\gamma} [S^{(1/2, j_1)}(Q)]^{\gamma\delta} [\mathcal{V}_\nu^{(1/2, j_1)}(Q, p)]_{\delta\beta}. \quad (6.5)$$

These amplitudes are shown in the Figs. 4,5,6. Their gauge invariance is ensured by the Ward-Takahashi identity (5.11) (c.f. [19]).

It is furthermore quite useful to define the following quantities:

$$\tilde{U}_{\mu\nu}^{(1/2, j_1)}(p', Q, p) = [\bar{f}^{(1/2, j_1)}(p')]^\alpha [U_{\mu\nu}^{(1/2, j_1)}(p', Q, p)]_{\alpha\beta} [f^{(1/2, j_1)}(p)]^\beta, \quad (6.6)$$

$$\tilde{\mathcal{C}}_{\mu\nu}^{(1/2, j_1)} = [\bar{f}^{(1/2, j_1)}(p')]^\alpha [\mathcal{C}_{\mu\nu}^{(1/2, j_1)} + \mathcal{C}_{\nu\mu}^{(1/2, j_1)}]_{\alpha\beta} [f^{(1/2, j_1)}(p)]^\beta, \quad (6.7)$$

with the $[f^{(1/2,j_1)}(p)]^\alpha$ matrices taken from (3.28) and (3.29).

In order to find the averaged square amplitude, we employ the following general expression:

$$\overline{|\mathcal{M}^{(1/2,j_1)}|^2} = \frac{1}{4} \sum_{\lambda,\lambda',\ell,\ell'} [\mathcal{M}^{(1/2,j_1)}][\mathcal{M}^{(1/2,j_1)}]^\dagger \quad (6.8)$$

$$= \text{Tr} \left[(\mathcal{M}^{(1/2,j_1)})_{\mu\nu}(p', Q, R, p) (\mathcal{M}^{(1/2,j_1)})^{\nu\mu}(p, R, Q, p') \right], \quad (6.9)$$

valid for any j_1 . Here, we have defined

$$\mathcal{M}_{\mu\nu}^{(1/2,j_1)}(p', Q, R, p) = \frac{e^2}{2} \left(\frac{\not{p}' + m}{2m} \right) \mathbb{U}_{\mu\nu}^{(1/2,j_1)}(p', Q, R, p), \quad (6.10)$$

$$\mathbb{U}_{\mu\nu}^{(1/2,j_1)}(p', Q, R, p) = \tilde{U}_{\mu\nu}^{(1/2,j_1)}(p', Q, p) + \tilde{U}_{\nu\mu}^{(1/2,j_1)}(p', R, p) - \tilde{\mathcal{C}}_{\mu\nu}^{(1/2,j_1)}. \quad (6.11)$$

Furthermore, \tilde{U} and $\tilde{\mathcal{C}}$ have been defined in (6.6), (6.7), and use has been made of the projector,

$$[\mathbb{P}_+^{(1/2,j_1)}(p)]_{\alpha\beta} = [f^{(1/2,j_1)}(p)]_\alpha \left(\frac{\not{p} + m}{2m} \right) [\bar{f}^{(1/2,j_1)}(p)]_\beta, \quad (6.12)$$

in combination with the projector on the photon polarization vectors,

$$\sum_{\ell} \epsilon^\mu(\mathbf{q}, \ell) [\epsilon^\nu(\mathbf{q}, \ell)]^* = -g^{\mu\nu}. \quad (6.13)$$

The result for each j_1 value can be expressed by a single general formula valid for any g_{s_i} , as:

$$\overline{|\mathcal{M}^{(1/2,j_1)}(g^{(1/2,j_1)})|^2} = f_0 + f_D + \frac{e^4(2m^2 - s - u)}{16m^2(m^2 - s)^2(m^2 - u)^2} \sum_{k=1}^4 (g_{(s_i)} - 2)^k a_k, \quad (6.14)$$

where:

$$a_1 = -32m^2(m^2 - s)(m^2 - u)(2m^2 - s - u), \quad (6.15)$$

$$a_2 = -4(13m^8 - 17(s + u)m^6 + (6(s^2 + u^2) + 20us)m^4 - 7su(s + u)m^2 + 3s^2u^2), \quad (6.16)$$

$$a_3 = -8(m^2 - s)^2(m^2 - u)^2, \quad (6.17)$$

$$a_4 = (m^2 - s)(m^2 - u)(m^2(s + u) - 2su), \quad (6.18)$$

and with f_0, f_D standing for,

$$f_0 = \frac{4e^4(5m^8 - 4(s + u)m^6 + (s^2 + u^2)m^4 + s^2u^2)}{(m^2 - s)^2(m^2 - u)^2}, \quad (6.19)$$

$$f_D = -\frac{2e^4(-2m^2 + s + u)^2}{(m^2 - s)(m^2 - u)}. \quad (6.20)$$

Here, s and u , are the standard Mandelstam variables.

A comment is due on the unspecified parity of the propagator, typical for all second order approaches. We here gain control over the issue through the systematic explicit specification of the parities of the external legs. Notice that none of the Feynman diagrams entering the calculation of the cross sections contains opposite parity states as external legs. At most, such a state could appear in the internal lines. However, as long as such diagrams would be equivalent to diagrams with a chirality operator inserted in each one of the vertexes, they will be indistinguishable from the regular diagrams with an internal line consistent with the initial state, and can be accounted for by a proper normalization. Moreover, we investigated whether Feynman diagrams with legs of opposite parities, i.e. such invoking an axial electromagnetic current in one of the vertexes, could contribute to the Compton scattering process under investigation and found them vanishing within the $(1/2, 0) \oplus (0, 1/2)$ sector for $g_s = 2$. In this way, the parity conservation is strictly respected by the spin-1/2⁺ Dirac particle, as it should be. The spin-1/2⁻,

and the (here omitted) spin- $3/2^-$ residents in $(1/2, 1) \oplus (1, 1/2)$, however, are different and such diagrams are non vanishing for all g_s values, certainly an interesting issue, worth being pursued in future research. For the time being, we limit ourselves to the observation that such fermions may be eligible as matter fields in dual theories of electromagnetism with co-existing axial and regular photons.

Obtaining now the differential cross-section in the laboratory frame from the squared amplitudes in (6.14) is straightforward (see [3] for technical details). After some algebraic manipulations one arrives at,

$$\frac{d\sigma(g_{(s_i)}, \eta, x)}{d\Omega} = \frac{|\mathcal{M}^{(1/2, j_1)}(g_{(s_i)})|^2}{4(4\pi m^2)} \frac{(\omega')^2}{\omega^2} = z_0 + z_D + \frac{(x-1)r_0^2}{64((x-1)\eta-1)^3} \sum_{k=1}^4 (g_{(s_i)} - 2)^k b_k, \quad (6.21)$$

where $r_0 = e^2/(4\pi m) = \alpha m$, $\eta = \omega/m$ with ω being the energy of the incident photon, ω' the energy of the scattered photon, while $x = \cos \theta$, θ giving the scattering angle in the laboratory frame. In (6.21) z_0 denotes the standard differential cross-section for Compton scattering off spin-0 particles and $(z_0 + z_D)$ is the standard differential cross-section for Compton scattering off Dirac particles, i.e.,

$$z_0 = \frac{(x^2 + 1)r_0^2}{2((x-1)\eta-1)^2}, \quad (6.22)$$

$$z_D = -\frac{(x-1)^2\eta^2 r_0^2}{2((x-1)\eta-1)^3}. \quad (6.23)$$

We further have introduced the following notations,

$$b_1 = -32(x-1)\eta^2, \quad (6.24)$$

$$b_2 = 4(x^2 - 3x + 8)\eta^2, \quad (6.25)$$

$$b_3 = 16\eta^2, \quad (6.26)$$

$$b_4 = (x+3)\eta^2. \quad (6.27)$$

The differential cross-section (6.21) is found to have the following limits:

$$\lim_{x \rightarrow 1} \frac{d\sigma(g_{(s_i)}, \eta, x)}{d\Omega} = r_0^2, \quad (6.28)$$

$$\lim_{\eta \rightarrow 0} \frac{d\sigma(g_{(s_i)}, \eta, x)}{d\Omega} = \frac{r_0^2}{2}(x^2 + 1), \quad (6.29)$$

$$\lim_{\eta \rightarrow \infty} \frac{d\sigma(g_{(s_i)}, \eta, x)}{d\Omega} = 0, \quad (6.30)$$

meaning that in the forward direction ($x = \cos \theta = 1$) it takes the r_0^2 value. In the classical $\eta \rightarrow 0$ limit the differential cross section is symmetric with respect to the scattering angle θ , while in the high energy $\eta \rightarrow \infty$ limit it vanishes independently of the $g_{(s_i)}$ factor value. This observation applies to each one of the two $j_1 = 0^-$, and $j_1 = 1$ sectors of ψ_μ considered here, and the related $g_{(1/2+)} = 2$ and $g_{(1/2-)} = -2/3$ values. The behavior of the differential cross-section is displayed in Fig. 7, which is a plot of,

$$d\tilde{\sigma}(g_{(s_i)}, \eta, x) \equiv \frac{1}{r_0^2} \frac{d\sigma(g_{(s_i)}, \eta, x)}{d\Omega}, \quad (6.31)$$

for the two different $g_{(1/2+)} = 2$, $g_{(1/2-)} = -2/3$ values of the gyromagnetic factors.

Integration of (6.21) over the solid angle leads to the total cross-sections,

$$\sigma(g_{(s_i)}, \eta) = s_0 + s_D + \sum_{k=1}^4 (g_{(1/2, j_1)} - 2)^k \left(\frac{c_k}{128\eta(2\eta+1)^2} + \frac{\log(2\eta+1)h_k}{256\eta^2} \right) 3\sigma_T, \quad (6.32)$$

where σ_T stands for the Thompson cross section $\sigma_T = (8/3)\pi r_0^2$. The following notations have been used,

$$s_0 = \frac{3(\eta+1)\sigma_T(2\eta(\eta+1) - (2\eta+1)\log(2\eta+1))}{4\eta^3(2\eta+1)}, \quad (6.33)$$

$$s_D = \frac{3\sigma_T((2\eta+1)^2\log(2\eta+1) - 2\eta(3\eta+1))}{8\eta(2\eta+1)^2}, \quad (6.34)$$

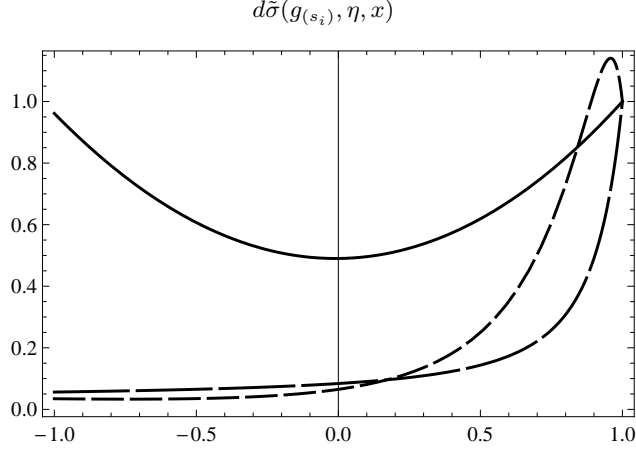


Fig. 7: Differential cross sections for particles in the $(1/2, j_1) \oplus (j_1, 1/2)$ sector of the four-vector spinor as a function of $x = \cos \theta$ (where θ is the scattering angle). The solid curve represents the classical limit, i.e. the differential cross section $d\tilde{\sigma}(g_{(s_i)}, \eta, x)$ from (6.31) at $\eta = \omega/m = 0$ (where ω is the energy of the incident photon), the long-dashed curve corresponds to $d\tilde{\sigma}(g_{(1/2+)}, \eta, x)$ with $g_{(1/2+)} = 2$ at the energy of $\eta = 4$, and the short-dashed curve represents $d\tilde{\sigma}(g_{(1/2-)}, \eta, x)$ with $g_{(1/2-)} = -2/3$ also at the energy of $\eta = 4$. The differential cross section has the correct Thompson limit for any $g_{(s_i)}$ value.

where s_0 and $(s_0 + s_D)$ are the standard cross-sections for Compton scattering off spin-0 and spin-1/2 Dirac particles, while the c and h coefficients stand for the following quantities,

$$c_1 = -32\eta(3\eta + 1), \quad (6.35)$$

$$c_2 = 4(6\eta^3 + \eta^2 + 8\eta + 3), \quad (6.36)$$

$$c_3 = 16\eta^3, \quad (6.37)$$

$$c_4 = \eta(4\eta^2 + 3\eta + 1), \quad (6.38)$$

$$h_1 = 32\eta, \quad (6.39)$$

$$h_2 = 4(\eta - 3), \quad (6.40)$$

$$h_3 = 0, \quad (6.41)$$

$$h_4 = -\eta. \quad (6.42)$$

The total cross section (6.32) has the following limits,

$$\lim_{\eta \rightarrow 0} \sigma(g_{(s_i)}, \eta) = \sigma_T, \quad (6.43)$$

$$\lim_{\eta \rightarrow \infty} \sigma(g_{(s_i)}, \eta) = \frac{3}{128}(g_{(s_i)} - 2)^2((g_{(s_i)})^2 + 2)\sigma_T. \quad (6.44)$$

Consequently, while in the $g_{(1/2+)} = 2$ case the scattering cross section for the genuine Dirac particle is asymptotically vanishing, for $g_{(1/2-)} = -2/3$, it approaches the fixed $\frac{11\sigma_T}{27}$ value. In Fig. 8 the following quantity is plotted,

$$\tilde{\sigma}(g_{(s_i)}, \eta) \equiv \frac{1}{\sigma_T} \sigma(g_{(s_i)}, \eta). \quad (6.45)$$

For $g_{(1/2+)} = 2$ one observes the usual decreasing behavior of the Dirac cross section with energy increase, while for $g_{(1/2-)} = -2/3$ i.e. for spin-1/2 in $(1/2, 1) \oplus (1, 1/2)$, the cross section $\tilde{\sigma}(g_{(s_i)}, \eta)$ at high energy approaches the fixed value of $\frac{11}{27}$ as one can see in the Fig. 8.

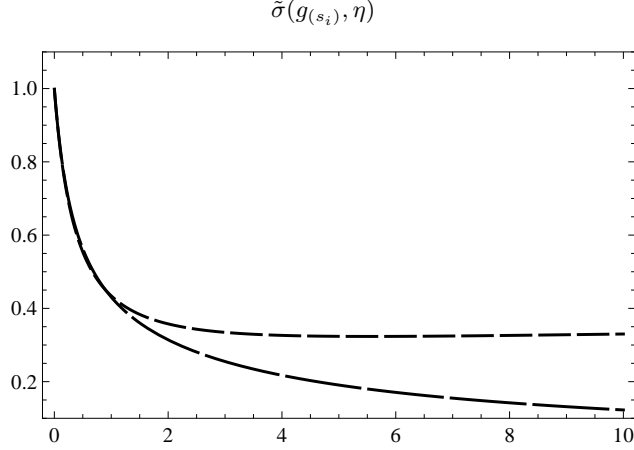


Fig. 8: The total cross sections $\tilde{\sigma}(g_{(s_i)}, \eta)$ in eq. (6.45) for particles in the $(1/2, j_1) \oplus (j_1, 1/2)$ sectors of the four-vector spinor as a function of $\eta = \omega/m$ (where ω is the energy of the incident photon) up to $\eta = 10$. The long-dashed curve corresponds to $\tilde{\sigma}(g_{(1/2^+)}, \eta)$ with $g_{(1/2^+)} = 2$, and the short-dashed curve represents $\tilde{\sigma}(g_{(1/2^-)}, \eta)$ with $g_{(1/2^-)} = -2/3$. While $\tilde{\sigma}(g_{(1/2^+)}, \eta)$ is vanishing in the ultra relativistic limit, $\tilde{\sigma}(g_{(1/2^-)}, \eta)$ approaches a fixed value, as visible from the equation (6.44), and also in accord with unitarity.

B. Calculation with the contracted equations

The calculation of the Compton scattering with the contracted equations and associated Lagrangians is in reality equivalent to the calculation of Compton scattering with the most general spin-1/2 equation (1.5), already studied earlier in ref. [3], and will not be repeated here. The result on the total cross section reported in the latter work completely and precisely coincides with our general formula for any g_s given in the above equation (6.44). Also the results on the Thompson limit (6.43) are fully identical. The revealed identity between the description of the spin-1/2⁻ sector in the four-vector spinor by means of the four gauged spinorial degrees of freedom described by the contracted equation (4.5) on the one side, with its description by means of the gauged 16 dimensional spin-1/2⁻ four-vector-spinors in the gauged Lagrangians in eq. (5.4), on the other side, allows us to conclude that within the framework suggested,

- the spin-1/2⁻ particle in the four-vector spinor, ψ_μ , indeed effectively behaves as a truly “quadratic” relativistic fermion,
- the gauging procedure conserves the number of the degrees of freedom, identified prior gauging at the free particle level by the Poincaré covariant spin projectors in (3.26),
- a spin-1/2 upon gauging, in the absence of gauged space-time symmetries and gauged covariant spin-projectors can still be identified through the number of its gauged degrees of freedom, which in our case was shown to be four, and thereby same as the number of the free degrees of freedom, correctly identified prior gauging by the aforementioned Poincaré covariant spin projector.

VII. CONCLUSIONS

In this work we studied the $so(1,3)$ irreducible spin-1/2 degrees of freedom within the $(1/2^+, 1/2^-, 3/2^-)$ triad in the four-vector spinor, ψ_μ , prior and upon gauging. We showed that both particles can be described by the generalized Feynman-Gell-Mann equation (1.5), which at the free particle level is equivalent to the Dirac equation. However, at the interacting level, the value of the gyromagnetic factor acquires importance and while for spin-1/2⁺ and $g_{(1/2^+)} = 2$ the quadratic equation allows for a bilinearization towards the Dirac equation and its conjugate, for spin-1/2⁻, characterized by $g_{(1/2^-)} = -2/3$ no linearization is possible.

The approach developed was formulated in reference to the $so(1,3)$ irreducible degrees of freedom in this space, and motivated by their interconnections by spin-up and down ladder operators, reviewed in

the Appendix I. It allowed for the description of all degrees of freedom in ψ_μ on equal footing and by means of fully relativistic, representation- and spin-specific wave equations, and associated Lagrangians, all being second order in the momenta. The $so(1, 3)$ irreducibility turned out to be crucial for the correct identification of the single spin- $1/2^+$ sector, $(1/2, 0) \oplus (0, 1/2)$, which we found identical to the Dirac particle and characterized by a gyromagnetic factor of $g_{(1/2^+)} = 2$, which allows for a bi-linearization. We furthermore reproduced the precise Compton scattering results off this target known from the Dirac theory. As a cross-check we also calculated the gyromagnetic factor for the $so(1, 3)$ reducible Clebsch-Gordan combination, $[\mathcal{U}^{\text{SS}}(\mathbf{p}, \frac{1}{2}, \lambda)]^\alpha$, in the above equation (2.2) and found a vanishing g_s value, however we did not include this calculation into the text for the sake of not overloading the presentation. Similarly, for the $so(1, 3)$ reducible spin- $1/2^-$ Clebsch-Gordan combinations, $[\mathcal{U}^{\text{VS}}(\mathbf{p}, \frac{1}{2}, \lambda)]^\alpha$ in (2.3), a $g_{(1/2^-)} = 1/3$ value has been calculated. To the amount Wigner's definition [14] prescribes that particles have to transform irreducibly under space-time symmetries, we conclude that the $so(1, 3)$ reducible degrees of freedom have to be discarded as unphysical. We recall that according to the standard definition by Wigner, fundamental particles have to transform according to unitary irreducible representation spaces of the Poincaré algebra. In field theory, such representations are built up as Fourier transforms of quantum states transforming according to $so(1, 3)$ irreducible finite dimensional non-unitary representation spaces of the Lorentz algebra. Finite-dimensional non-unitary representation spaces are of key importance in the description of spin degrees of freedom and their space-time transformation properties are sufficient to define at the free particle level a wave equation and corresponding Lagrangian in terms of the corresponding algebra elements and invariants.[18]

Second order fermion theories have attracted recently attention through their applications within the world line formalism in Yang-Mills theories [20]–[23]. However, these are theories based upon two-component spinors, while our framework is based on full flashed Lorentz invariant four-vectors and four-spinors. The information on the complete dimensionality of the relativistic representation spaces is fully encoded by our respective wave equations and Lagrangians, an issue we discussed after the equations (3.37)–(3.44). The quadratic Lagrangians put at work in the evaluation of the Compton scattering amplitudes are such that prior gauging the spin- $1/2^-$ - $3/2^-$ degrees of freedom in the $so(1, 3)$ irreducible $(1/2, 1) \oplus (1, 1/2)$ sector of ψ_μ have been neatly separated without auxiliary conditions at the free particle level by means of Poincaré covariant mass and spin -projectors based on the squared Pauli-Lubanski-vector operator, a Casimir invariant of the Poincaré group. We brought an argument in support of the statement that this separation remained respected by the gauging procedure. The argument was based on the observation that the gauged spin- $1/2^-$ Lagrangian within the full 16 dimensional space happened to provide equivalent Compton scattering description as the 4 dimensional spinor solving the generalized Feynman–Gell-Mann equation in (1.5), and for $g_{(1/2^-)} = -2/3$. To the amount the equation (1.5) is bi-linearizable into gauged Dirac equations exclusively for a gyromagnetic factor taking the special value of two units, no linearization for spin- $1/2^-$ is possible and this sector from the four-vector spinor behaves as a truly “quadratic”, fully relativistic fermion. As a further consequence of the observed equality we wish to note that along the line of ref. [12], it is easy to verify that the gauged degrees of freedom in the generalized Feynman–Gell-Mann equation (1.5) propagate causally within an electromagnetic environment. We expect our findings to contribute to the understanding of the properties of the particles residing within the four-vector spinor space and shed more light on possible interferences between the spin sectors in physical processes governed by the spin-up and down ladder operators discussed in the Appendix I.

Appendix I: Spin-ladder operators within $(1/2, 1) \oplus (1, 1/2)$

From group theory it is well known that the states spanning any irreducible representation space are related to each other by ladder operators. Specifically, the degrees of freedom within representation spaces of multiple spins are interconnected by spin-up and down ladder operators. We here are interested in such operators within the two-spin valued space, $(1/2, 1) \oplus (1, 1/2)$ and begin designing them in the rest frame, $p_0 = m$, $\mathbf{p} = \mathbf{0}$, where their form is specifically simple and given by,

$$\mathcal{K}_\pm^P(\mathbf{0}) = \pm \frac{1}{\sqrt{2}} (J_-^V \otimes J_+^S - J_+^V \otimes J_-^S). \quad (7.1)$$

Here the label P on \mathcal{K}_\pm^P indicates that this operator ladders only between eigenstates to the Poincaré projectors. Here J_\pm^V and J_\pm^S are in turn the conventional spin-up and down ladder operators in the respective four-vector $(1/2, 1/2)$, and the Dirac- spinor, $(1/2, 0) \oplus (0, 1/2)$, representations,

$$J_\pm^{V/S} = J_1^{V/S} \pm iJ_2^{V/S}. \quad (7.2)$$

The rest-frame four-vector-spinors, $w_\pm^{(1/2,1)}(\mathbf{0}, s_i, \lambda)$, with $s_1 = 1/2$, and $s_2 = 3/2$, belong to the $(1/2, 1) \oplus (1, 1/2)$ sector and are,

$$w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{1}{2}, \lambda\right) = -2\mathcal{P}_F^{(1/2,1)}\mathcal{U}_\pm\left(\mathbf{0}, \frac{1}{2}, \lambda\right), \quad (7.3)$$

$$w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{3}{2}, \sigma\right) = -\mathcal{P}_F^{(1/2,1)}\mathcal{U}_\pm\left(\mathbf{0}, \frac{3}{2}, \sigma\right), \quad (7.4)$$

where the (-2) and (-1) factors have been additionally introduced for the sake of their normalization to ± 1 in dependence on their parities. One can prove commutativity of the Poincaré ladder operators, $\mathcal{K}_\pm^P(\mathbf{0})$, and the Lorentz projector $\mathcal{P}_F^{(1/2,1)}$, which can be used to build up Lorentz ladder operators among these states as,

$$\mathcal{K}_\pm^L(\mathbf{0}) = 2\mathcal{P}_F^{(1/2,1)}\mathcal{K}_\pm^P(\mathbf{0}). \quad (7.5)$$

The above operator annihilates spin-3/2 with spin-projections $\lambda_i = \pm 3/2$,

$$\begin{aligned} \mathcal{K}_\pm^L(\mathbf{0})w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{3}{2}, \pm\frac{3}{2}\right) &= 2\mathcal{P}_F^{(1/2,1)}\mathcal{K}_\pm^P(\mathbf{0})w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{3}{2}, \pm\frac{3}{2}\right) \\ &= -2\mathcal{P}_F^{(1/2,1)}\mathcal{K}_\pm^P(\mathbf{0})\mathcal{U}_\pm\left(\mathbf{0}, \frac{3}{2}, \pm\frac{3}{2}\right) = 0. \end{aligned} \quad (7.6)$$

Instead, while acting on spin-3/2 with spin-projection $\lambda = \pm 1/2$, a spin-1/2 state is reached according to,

$$\mathcal{K}_-^L(\mathbf{0})w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{3}{2}, \pm\frac{1}{2}\right) = 2\mathcal{P}_F^{(1/2,1)}\mathcal{K}_-^P(\mathbf{0})w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{3}{2}, \pm\frac{1}{2}\right), \quad (7.7)$$

$$= -2\mathcal{P}_F^{(1/2,1)}\mathcal{K}_-^P(\mathbf{0})\mathcal{U}_\pm\left(\mathbf{0}, \frac{3}{2}, \pm\frac{1}{2}\right) \quad (7.8)$$

$$= -2\mathcal{P}_F^{(1/2,1)}\mathcal{U}_\pm\left(\mathbf{0}, \frac{1}{2}, \pm\frac{1}{2}\right). \quad (7.9)$$

In effect, the spin-3/2 is lowered by one unit down to spin-1/2,

$$\mathcal{K}_-^L(\mathbf{0})w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{3}{2}, \pm\frac{1}{2}\right) = w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{1}{2}, \pm\frac{1}{2}\right), \quad (7.10)$$

and vice verse. One can also rise the spin-1/2 to spin-3/2 as visible from,

$$\mathcal{K}_+^L(\mathbf{0})w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{1}{2}, \pm\frac{1}{2}\right) = w_\pm^{(1/2,1)}\left(\mathbf{0}, \frac{3}{2}, \pm\frac{1}{2}\right), \quad (7.11)$$

thus confirming $\mathcal{K}_\pm^L(\mathbf{0})$ as the rest-frame spin up and down ladder operators. The covariant spin-ladder operators in any inertial frame are then obtained by similarity transforming $\mathcal{K}_\pm^L(\mathbf{0})$ by the boost operator in the representation under consideration,

$$\mathcal{K}_\pm^L(\mathbf{p}) = B(\mathbf{p})\mathcal{K}_\pm^L(\mathbf{0})B^{-1}(\mathbf{p}) = B(\mathbf{p})\mathcal{K}_\pm^L(\mathbf{0})B(-\mathbf{p}). \quad (7.12)$$

The above considerations show that one can choose anyone of the spin degrees of freedom in $(1/2, 1) \oplus (1, 1/2)$ as a point of departure for building up the other one. These facts force us to conclude that there is no kinematic reason, and as we have shown there is also no dynamical reason for which one type of degrees of freedom in a representation space of multiple spins, has to be preferred over another type.

Appendix II: Spin-3/2 within the four-vector-spinor

A. The rigorous spin-3/2 third order equation, its problems and the need for lowering its order

The wave equations for the spin-3/2 four-vector spinors from (2.4)-(2.7), $[\mathcal{U}_+(\mathbf{p}, \frac{3}{2}, \sigma)]^\alpha$, and $[\mathcal{U}_-(\mathbf{p}, \frac{3}{2}, \sigma)]^\alpha$, of positive and negative parity, respectively, are defined through,

$$\left[\Pi_\pm^{(3/2)}(p)\right]^{\alpha\beta} \left[\mathcal{U}_\pm(\mathbf{p}, \frac{3}{2}, \sigma)\right]_\beta = \left[\mathcal{U}_\pm(\mathbf{p}, \frac{3}{2}, \sigma)\right]^\alpha, \quad (7.13)$$

where $\left[\Pi_\pm^{(3/2)}(p)\right]^{\alpha\beta}$ denotes a covariant projector on these vector-spinors which is given by,

$$\left[\Pi_\pm^{(3/2)}(p)\right]^{\alpha\beta} = \frac{\mp \not{p} + m}{2m} \left[\mathbb{P}^{(3/2)}(p)\right]^{\alpha\beta}. \quad (7.14)$$

Here $\left[\mathbb{P}^{(3/2)}(p)\right]^{\alpha\beta}$ stands for the covariant spin-3/2 projector [15] defined in terms of the squared Pauli-Lubanski operator from (2.13)-(2.14) as,

$$\left[\mathbb{P}^{(3/2)}(p)\right]_{\alpha\beta} = -\frac{1}{3} \left(\frac{\mathcal{W}^2(p)_{\alpha\beta}}{p^2} + \frac{3}{4} g_{\alpha\beta} \right), \quad (7.15)$$

while $(\pm \not{p} + m)/(2m)$ is the covariant parity projector in the Dirac spinor space. The projector $\left[\Pi_\pm^{(3/2)}(p)\right]^{\alpha\beta}$ leads to a wave equation which is third order in the momenta. However, differential equations of an order higher than two are both conceptually and technically problematic. The issue is that they lead to higher-order field theories which are plagued by severe inconsistencies such as ghost solutions of the bad type, kinetic terms of wrong signs, states of negative norms, and which violate unitarity. Moreover, particles in such theories can suffer non local propagation. The experimentally established fundamental theories in contemporary physics are all based on second order Lagrangians, the Standard Model being a prominent example. Only in effective theories with over integrated fields, higher order Lagrangians may appear. At the root of the troubles related to higher order differential equations is the so called Ostrogradskian instability [24] which, at the level of, say, classical mechanics, for concreteness, predicts phase spaces of unstable orbits. In order to circumvent such problems, it is desirable to lower the order of the above equation (7.13) before employing it in physical processes. Such a lowering can be carried out in two steps, first to second-, and then to first order.

B. From third order equation to a quadratic

A second order equation can be obtained by replacing in (7.14) the parity operator by the mass projector, p^2/m^2 , according to,

$$\frac{\mp \not{p} + m}{2m} \longrightarrow \frac{p^2}{m^2}, \quad (7.16)$$

in which case (7.15) becomes,

$$-\frac{p^2}{m^2} \frac{1}{3} \left(\frac{\mathcal{W}^2(p)_{\alpha\beta}}{p^2} + \frac{3}{4} g_{\alpha\beta} \right) \left[\mathcal{U}_\pm(\mathbf{p}, \frac{3}{2}, \sigma)\right]^\beta = \left[\mathcal{U}_\pm(\mathbf{p}, \frac{3}{2}, \sigma)\right]_\alpha. \quad (7.17)$$

This path has been taken by the Ref. [4] and has been further pursued in [19] and [18]. The explicit wave equation resulting from (7.17), reported in [4] in all necessary technical detail, which we here bring for completeness in its most interesting electromagnetically gauged form, reads

$$\begin{aligned} \left((\pi^2 - m^2) g_{\alpha\beta} - i g_{(3/2)} \left(\frac{\sigma_{\mu\nu} [\pi^\mu, \pi^\nu]}{4} g_{\alpha\beta} - e F_{\alpha\beta} \right) + \frac{1}{3} (\gamma_\alpha \not{\pi} - 4\pi_\alpha) \pi_\beta \right. \\ \left. + \frac{1}{3} (\pi_\alpha \not{\pi} - \gamma_\alpha \pi^2) \gamma_\beta \right) \left[\mathcal{U}_\pm(\mathbf{p}, \frac{3}{2}, \sigma)\right]^\beta = 0. \end{aligned} \quad (7.18)$$

Here, π_μ is the minimally gauged four-momentum, $\pi_\mu = p_\mu - eA_\mu$ in (3.45). The spin-3/2 gyromagnetic factor, $g_{(3/2)}$, has been fixed to $g_{(3/2)} = 2$ from the requirement on causal propagation within an electromagnetic background, without that this equation is linearizable. In this way, it was demonstrated in [4] that the above second order spin-3/2 wave equation is free from the Velo-Zwanziger problem suffered by the linear Rarita-Schwinger framework. This value has been further examined in [19] and shown to lead to finite Compton scattering differential cross sections in forward direction and in accord with the desired unitarity. It has to be emphasized that couplings of the type, $F^{\mu\nu}\sigma_{\mu\nu}$ arise within the second order formalism upon minimal gauge of the momenta in the $so(1,3)$ algebraically prescribed $[\partial^\mu, \partial^\nu]$ commutators of the type first seen in eq. (1.11) and are not introduced *ad hoc* as non-minimal couplings as occasionally done within the Rarita-Schwinger framework [25] for the sake of getting rid of the unitarity problem.

C. From quadratic equation to a linear

Below we show how the linear Rarita-Schwinger framework relates to (7.17). Towards this goal we first recall that within the direct-product space $(1/2, 1/2) \otimes [(1/2, 0) \oplus (0, 1/2)]$ of interest here, $\mathcal{W}_\mu(p)$ expresses as the direct sum of the Pauli-Lubanski vectors, $W_\mu(p)$, and $\omega_\mu(p)$, in the respective $(1/2, 1/2)$ - and Dirac-building blocks according to (2.12). With the aid of the formulas given in this equation it is straightforward to calculate $\mathcal{W}_\mu^2(p)$ as,

$$[\mathcal{W}^2(p)]_\alpha^\beta = \omega^2(p)g_\alpha^\beta + [W^2(p)]_\alpha^\beta + 2(W^\mu(p))_\alpha^\beta \omega_\mu(p), \quad (7.19a)$$

$$\omega^2(p) = -\frac{1}{4}\sigma_{\lambda\mu}\sigma^\lambda{}_\nu p^\mu p^\nu = -\frac{3}{4}p^2 + \frac{1}{4}m\gamma_\eta\gamma_\lambda [p^\lambda, p^\eta], \quad (7.19b)$$

$$[W^2(p)]_\alpha^\beta = -2(g_\alpha^\beta g_{\mu\nu} - g_{\alpha\nu}g_\mu^\beta)p^\mu p^\nu = -2(g_\alpha^\beta p^2 - p_\alpha p^\beta), \quad (7.19c)$$

$$2(W^\mu(p))_\alpha^\beta \omega_\mu(p) = i\epsilon^\mu{}_\alpha{}^\beta{}_\sigma p^\sigma \gamma_5(p_\mu - \gamma_\mu \not{p}) = -i\epsilon^\mu{}_\alpha{}^\beta{}_\sigma p^\sigma \gamma_5 \gamma_\mu \not{p}, \quad (7.19d)$$

where we again for the sake of simplicity suppressed the spinorial indexes. In effect, the $\mathcal{W}^2(p)$ eigenvalue problem takes the following form,

$$(\mathcal{W}^2(p))_{\alpha\beta} \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \sigma \right) \right]^\beta = -\frac{11}{4}p^2 \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \sigma \right) \right]_\alpha + 2(\omega_\mu(p)W^\mu(p))_{\alpha\beta} \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \sigma \right) \right]^\beta. \quad (7.20)$$

In order to ensure projection on spin-3/2 by means of (7.17), i.e. in order to guarantee that the $\mathcal{W}^2(p)$ eigenvalue, $s_2(s_2 + 1)$ in (2.15) equals, $(-p^2 \frac{3}{2} (\frac{3}{2} + 1))$, it is necessary that the interference terms contributes $(-p^2)$ to the eigenvalue and satisfies,

$$2(\omega_\mu(p)W^\mu(p))_{\alpha\beta} \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \sigma \right) \right]^\beta = -p^2 \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \sigma \right) \right]_\alpha. \quad (7.21)$$

Substitution of (7.19d) into (7.20) and setting $\not{p} [\mathcal{U}_\pm(\mathbf{p}, 3/2, \sigma)]_\alpha = \mp m [\mathcal{U}_\pm(\mathbf{p}, 3/2, \sigma)]_\alpha$, amounts to

$$\left(\mp i\epsilon^\mu{}_\alpha{}^\beta{}_\sigma p^\sigma \gamma_5 \gamma_\mu - mg_\alpha{}^\beta + 2\frac{p_\alpha p^\beta}{m} \right) \left[\mathcal{U}_\pm \left(\mathbf{p}, \frac{3}{2}, \sigma \right) \right]_\beta = 0. \quad (7.22)$$

The latter equation coincides, modulo the legitimate replacement, $2p^\alpha p^\beta / m \rightarrow m\gamma^\alpha \gamma^\beta$, allowed by the auxiliary conditions, $p \cdot \mathcal{U}_\pm(\mathbf{p}, 3/2, \lambda) = \gamma \cdot \mathcal{U}_\pm(\mathbf{p}, 3/2, \lambda) = 0$, with a text-book equation that represents in a compressed form the Rarita-Schwinger framework in the equations (2.24), (2.21), and (2.22) and reported among others in [10]. The above considerations relate the Rarita-Schwinger framework to a restriction of the covariant mass- m and spin-3/2 projector based wave equation in (7.18) down to a linear operator. Notice that as long as the spin-3/2 exclusively belongs to the $so(1,3)$ irreducible sector of the four-vector spinor, the application of the Lorentz projector, $\mathcal{P}_F^{(1/2,1)}$ from (3.9) is superfluous. The formalism systematically employed in the present work is consistent with the second order equation (7.17)

from above.

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